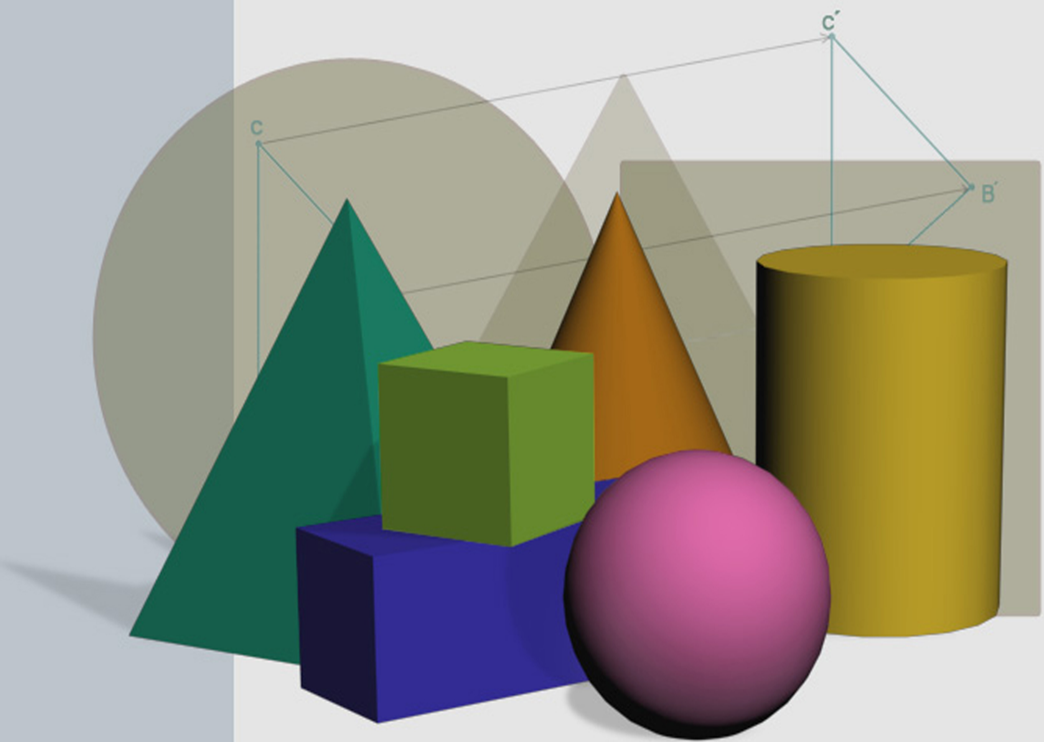
DIDACTICS OF GEOMETRY

FOR ELEMENTARY SCHOOLS



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"Knowing Geometry is more than recognizing figures and bodies by their names: it is solving geometric problems based on known properties of figures and bodies; in situations that are generally intramathematical, geometric and with or without graphic support. Their solution is what gives meaning to the teaching of Geometry".

(Bronzina et al., 2009)

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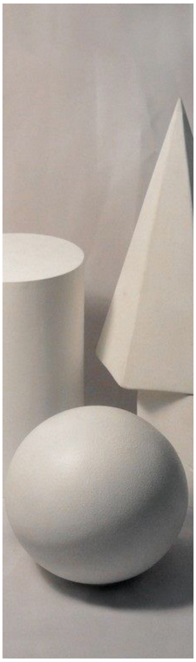

FOREWORD

The teaching of geometry plays important functions within the teaching-learning process of Mathematics in Elementary Education, since its study not only contributes to the development of geometric skills, but also has a considerable impact on the development of intellectual skills in schoolchildren from an early age, which will be applied by them at different times of their lives.

However, the teaching of this part of mathematics, which contributes to the development of thinking operations and creativity in schoolchildren, is paid unfavorable attention nowadays by many teachers. One of the main difficulties in its teaching-learning process takes place in Elementary Education, the one related to the development of skills. For this reason, this book focuses its study on the treatment of geometric contents and on the process of geometric skills development in Elementary Education.

The first chapter is dedicated to showing the teachers of Elementary Education, the main theoretical-methodological conceptions of the teaching-learning process of Mathematics in Education, highlighting the trends in the teaching-learning process, and its characteristics in the Cuban elementary school.

In the second chapter, we present an analysis of the geometric skills that should be developed by their students during their transit through Elementary Education. On the other hand, we present the theoretical foundations that support their development: principles, levels, indicators, actions and operations to guide and evaluate this process.

The principles determined favor conceiving the teaching-learning process of geometry in Elementary Education in a scientific manner, in accordance to the demands of the Cuban elementary school; while the proposed internal structure of geometric skills, together with the levels and indicators, allow guiding the teaching-learning process of geometry, according to the particularities of each schoolchild.

In the last chapter, we have decided to offer teachers methodological recommendations for the treatment of the fundamental geometric objects that are addressed in

Elementary Education. We have considered it necessary to separate these suggestions by cycle, due to the way in which the contents are treated in each one of them.

We do not think it is fair to conclude the book without first providing teachers with the procedure to create some teaching aids for the treatment of geometric contents and the development of skills in Elementary Education. We hope that these will be of great support in the teaching-learning process, and that they may motivate teachers to create their own teaching aids. Students will be very grateful to them if they do so.

At the end of the text we have added the list of sources consulted, which may be of interest to teachers to continue studying the topic. We hope that by the end of the reading, teachers will fulfill all their expectations. It is our greatest hope that the book will be useful in self-preparation, and that teachers will be able to put the ideas presented into practice in the teaching-learning process of geometry in Elementary Education.

The authors

01.

THE TEACHING-LEARNING PROCESS OF GEOMETRY IN ELEMENTARY EDUCATION

"The teaching of geometry must be a central core of the school curriculum".

(Alsina Catalá et al.,1989)

1.1. Theoretical-Methodological Conceptions of the Teaching-Learning Process of Mathematics in Elementary Education

In the international community of mathematics educators, the term "*Mathematics Education*" is frequently used, according to Díaz Godino (2010), as a synonym for "*Didactics of Mathematics*", to refer to the scientific discipline, which from a theoretical and practical point of view, studies the problems that arise in the teaching-learning process of Mathematics and proposes new theories for its transformation.

Mathematics Education as a social, heterogeneous and complex system (Díaz Godino, 2010) has among its components or fields: the practical and reflexive action on the teaching-learning process of mathematics; the technology that is proposed to be elaborated and used, along with the rest of the materials and resources; and scientific research, which analyzes the functioning of the process as a whole.

In Mathematics Education, according to Castillo, et al. (2006), three perspectives can be identified: the constructivist one, whose main ideas come from Piaget's writings; the sociocultural perspective, whose theoretical framework is based on Vygotsky's ideas; and the interactionist perspective, which is based on Blumer's ideas. However, they complement one another when they aim at a Mathematics Education focused on the acquisition of capacities, skills, and values that allow individuals to constantly update their knowledge in order to apply it to the world around them.

Mathematics Education, from the constructivist perspective, emphasizes how mathematical knowledge is constructed, through the relationship of the subject with the environment and the organization of his mental actions; from the sociocultural perspective, it is considered a social process, which transits, in the activity, from the inter-psychological to the intra-psychological, supported by mediators and instruments; -while from the interactionist perspective, from a socio-constructivist point of view, emphasis is placed on both individual and social processes, through participation and negotiation.

D'Ambrosio (2005), when looking to the future, with respect to Mathematics Education, recognizes how it is moving towards integration with the rest of the areas of knowledge, mainly in the most developed countries with a strong mathematical tradition and growing economy; achieved, initially, from the intra-material relationship.

From a pedagogical point of view, mathematics education in elementary school values the role of sensation and perception as the basis of mathematical knowledge, and the possibilities of its use in the interpretation, understanding and explanation of the social and historical context, which becomes the basis of their cognitive activity. Such consideration highlights the formative orientation of the activities that are organized, from the curriculum, for Mathematics Education; which should be adjusted to the characteristics of the student, the educational level, and the nature of the mathematical content.

In this context, the principles stated by Ruiz de Ugarrio (1965), for the teaching of Mathematics in Elementary Education, emphasize the importance of representations, in the transition from perception to abstract thinking; the role of comprehension, reflection and language development, in rational activity; and practice, as the beginning and end of all cognitive activity, since it is from it that the individual applies in life what he has learned as a result of his thinking.

In the direction of the teaching-learning process of mathematics at the international level, since the 1990s, there has been an influence of the curricular guidelines developed by the American Council of Teachers of Mathematics (National Council of Teachers of Mathematics, 2000), known as "*Principles and Standards for School Mathematics*". These principles do not refer to specific mathematical content or processes; they only describe aspects related to programs and influence curriculum development, lesson planning, assessment design, among other issues.

Among its precepts are the need to achieve equality in mathematics education, high expectations and support for all students; a coherent curriculum focused on mathematics at all educational levels; teaching that requires an understanding of students' prior knowledge and the new content they need to learn; learning that encourages the active construction of mathematical content; assessment to support mathematics learning, to provide useful information to both teachers and students; and to consider technology as an essential resource to stimulate learning.

In line with these conceptions, the use of the problem-solving teaching method in the treatment of mathematical content (National Council of Teachers of Mathematics, 2000), was reinforced from different and interrelated points of view: teaching to solve problems, about problem-solving and via problem-solving.

The first one proposes that schoolchildren solve problems that promote research and their applications in life; the second one to use heuristics in teaching, so that they learn to use solution strategies; and the third one to teach mathematics through problems and develop reasoning skills.

This situation allowed teachers and researchers to incorporate this method into their practices, based on contextualized proposals and methodologies in which they intended to transmit heuristic strategies, suitable for solving problems in general, to their students. Among the proposals, the one proposed by De Guzmán (1993), stands out, based on the ideas of Polya and Schoenfeld, in which Mathematics Education is conceived as a process of enculturation, with a variety of ways of approaching its teaching. This method emphasizes the importance of the schoolchild being able to manipulate mathematical objects (fundamental concepts), activate the mental layer and exercise creativity.

De Guzman (1993), also considers that the student should select the appropriate strategy; then reflect on the process of solving the problem, in order to improve it consciously, and make transfers from the activities performed to other aspects of mental work; develop self-confidence; have fun from the mental activity performed; get involved in the solution of other problems of science, of everyday life, and prepare for the challenges of technology.

Díaz Godino & Batanero (2009), recognize a fundamental role for the use of the problem-solving teaching method, as a function of the development of competencies in mathematical activity, taking into account some aspects of the ontosemiotic approach. From their point of view, the teacher must select and rework mathematical problems suitable for the students, using the appropriate resources; define, state and justify the concepts and procedures, taking into account previous notions; implement didactic figurations (system of actions between teacher and students), which allow optimizing learning; recognize the system of social and disciplinary norms that make possible the development of the process; to know the theoretical contributions made to Mathematics Education; to value the didactic suitability of the study processes planned or implemented in its different dimensions (epistemic, cognitive, affective, interactional, mediational and ecological); and to develop a positive attitude towards the teaching of mathematics, to value its formative role as well as its usefulness.

From these conceptions, we infer the need for teachers to develop the teaching-learning process of mathematics with a strategic approach, which helps students, with autonomy and self-determination, acquire the knowledge and skills necessary in their social and cultural context, in order to influence and transform it.

The predominance of the problem-solving teaching method and the ontosemiotic approach in the teaching-learning process of Mathematics, from the early grades, supports some theories of the French School of Didactics of Mathematics, as is the case of didactic situations, by Brousseau (1986), cited by Panizza (2003); and that of didactic transposition; formulated by Chevallard (2005).

Didactic situations allow the subject to construct mathematical knowledge in a context that is problematic for him/her. Then, according to the typology recognized by Díaz Godin et al. (2004), they are based on real problems that motivate and attract the attention of schoolchildren. In their resolution, they should offer the opportunity to investigate possible solutions, individually or in groups; favor the development of mathematical language; prove and demonstrate that the solution reached is correct; and use the knowledge acquired in common.

Chevallard's theory of didactic transposition (2005), explains the way in which mathematical contents (known knowledge) are transformed into contents that can be taught to students (knowledge to be taught) and put into practice in teaching (knowledge taught).

From the above, it is assumed that in this process the teacher performs actions such as simplification, modification and reduction of the complexity of the original knowledge, according to the particularities of each student and the social context in which the teaching-learning process takes place.

An important element to take into account is the statement made by Arrieta (1998), quoted by Barcia Martínez (2000), who refers to the fact that in several international congresses on educational mathematics it has been stated, in the teaching-learning process, that each school stage has its own rigor; the continuous support in concreteness, in reality; attending to the history of science and respecting it; extending

Mathematics Education as a "*know-how*"; and emphasizing the importance of induction and empiricism.

To these conceptions are added the contributions of Krutetskii (1968), mentioned by Wielewski (2005); and Giorgion (2010), who proposes a classification of cognitive styles, as references to establish levels of development of mathematical learning, related to the components: verbal-logical and visual-pictorial; which implies using concrete objects as support of the teaching-learning process, for the development of language and logical thinking.

Viewing Mathematics Education from these three perspectives confirms the need to integrate the essential aspects of a historical-cultural conception of the teaching-learning process. In this sense, external activity is identified as the starting point for the internalization of knowledge and skills; the relationship between learning and development; the role played by mediators as guides in this process; and the place given to language in the process.

This position takes on an essential meaning when it is assumed as a conception to support the teaching-learning process of Mathematics in the Cuban elementary school, based on the principles, laws and categories of general didactics; concretized in its school model (Rico Montero et al., 2008), which constitutes the basis for the achievement of developmental learning, since it tries to promote (Castellanos Simons et al., 2002) the integral development of the learner's personality; activate the appropriation of knowledge, skills and intellectual capacities; guarantee the unity of the cognitive and the affective-valuable in learners; promote the progressive transition from dependence to independence, as well as the development of the capacity to know, control and creatively transform reality; carry out lifelong learning, based on the mastery of skills for learning to learn and the need for constant self-education.

The model assumes as a reference Vygotsky's historical-cultural approach (1979), and the Cuban pedagogical traditions, from which a humanistic conception is configured, which gives value to the role of the subject in the active, direct and committed participation of his own personal and social growth.

From the contributions of these researchers, together with those of Puig Unzueta (2003), the renewal of the teaching-learning process of mathematics is projected; establishing as a reference the theory of cognitive performance levels, derived from learning assessment theories. This concept is closely related to the levels of assimilation, since the teaching-learning process is considered as the way through which the appropriation of knowledge and skills is achieved.

Performance levels have a systemic character, they allow to make the control of the whole teaching-learning process more dynamic, measure its quality and reorient it from the results achieved to the desired ones. In the same way, by evaluating the quality of the knowledge and skills of the students, they make it possible to establish different hierarchies and obtain a differentiating, flexible and diverse cognitive process.

Within the precepts of the elementary school model in Cuba (Rico Montero et al., 2008), it is emphasized that the levels of cognitive performance are useful to know the development achieved by the students; but when the teacher is going to plan a class, he/she should also be guided by the objectives and the three levels of assimilation.

The essence of these conceptions lies in the thesis of Marxist-Leninist psychology that all the psychic qualities of man are developed through the relationship of the subject with the historical-social-cultural reality and in the unity of the concrete and the abstract, as a requirement; this relationship constitutes an important element in the direction of the teaching-learning process of Mathematics, from the first grades of Elementary Education, since they recognize the role of the teacher for the orientation of the contents and that the students can apply them in practice.

1.2. Trends in the Teaching-Learning Process of Geometry in Elementary Education in Other Countries

The theoretical-methodological conceptions of the teaching-learning process of Mathematics in Elementary Education, in the last decades, have generated trends that serve as a basis for developing transformations in the methodological treatment of geometry, from the first grades of Elementary Education; which Proenza Garrido (2002), agrees with, when he points out that the teaching of geometry has been influenced by didactic models, spread in several countries; directed, some of them, to favor the development of specific geometric skills in schoolchildren.

Among the most important didactic models in the teaching of geometry is that of Van Hiele, proposed by Pierre Marie Van Hiele and Dina Van Hiele-Geldof in 1957, based on their teaching experience, Piaget's studies and Gestalt psychology. According to its authors, the purpose of the model is to develop geometric understanding or reasoning (insight) in schoolchildren. For this reason, the levels, as well as the phases of learning, are oriented to achieve this purpose.

The five levels (recognition, analysis, informal deduction, formal deduction and rigor) explain the way in which geometric comprehension occurs in schoolchildren, from intuitive to deductive forms of thought. It is not possible to alter the order of acquisition of the levels; each one serves as a basis for the next one, has its specific language associated with it; and its passage through them depends more on the instruction received by the student than on biological maturation.

Each level of understanding is structured in five phases of learning (information, directed orientation, explicitness, free orientation and integration), which allow the teacher to organize the teaching-learning process of geometry, so that schoolchildren acquire basic geometric knowledge, and then focus the activity on using and combining it. Van Hiele (1957), assures that in order for each schoolchild to advance in the active construction of knowledge and reach a higher level of understanding, he/she must overcome these phases.

This model can be used to guide the understanding of schoolchildren in Elementary Education, since the intuitive forms of geometric thinking begin to develop in the first grades up to the third level and continue their ascent in the rest of the teachings of the educational system; however, a disadvantage it presents is that it gives too much importance to the role of the instructional processes for the development of thinking, instead of the integral development of the schoolchild's personality, based on the unity that exists between the biological, social and cultural.

Jaime Pastor & Gutiérrez Rodríguez (1996), affirm that the studies initiated by Van Hiele (1957), have given rise to new research aimed at confirming whether these levels, in schoolchildren, describe geometric thinking exactly; their distribution, simultaneously, in two of them; the globality in all geometric concepts; the hierarchy, sequentiality, and the one that predominates most in the teaching-learning process; the unique existence of these five; and other research, related to the application of this theory in different aspects of geometry, such as Mathematics; as well as the combination with some that explain the way in which the understanding of geometric concepts occurs and develops.

It is in this last case that, according to Proenza Garrido (2002), some didactic models have promoted a trend in the teaching of geometry, such as the one of spatial location, proposed by Saiz (1997), where situations are proposed in which spatial actions are necessary in the environment; that of learning about space, proposed by Bishop (1997), which insists on showing that the spatial geometric ideas taught in school are not alien to the real world; that of geometric manipulations, proposed by Brenes (1997), which emphasizes the importance of using geometric figures and geometric bodies to develop spatial perception and better understanding of the world; and that of geometric manipulations, proposed by Brenes (1997), which emphasizes the importance of using geometric figures and bodies to develop spatial perception and better understanding of the world; that of geometric manipulations, presented by Brenes (1997), which emphasizes the importance of using geometric figures and bodies to develop spatial perception and a better understanding of the world; and that of the use of concrete materials, formulated by Castro (1997), which consists of using geometric models for the construction of basic geometric concepts.

In studies carried out by researchers, it is evident as a tendency, in countries such as Italy, Spain, Panama, and Venezuela that the teaching-learning process of geometry begins in the first grades of Elementary Education and extends throughout general education; the theory of Piaget, Vygotsky and Ausubel, as a psychological foundation, influences its treatment.

It was also noted that there are differences in opinions regarding the teaching of geometry. In some of these nations, it is taught as a separate subject, at the end of the rest of the mathematical content; and in others, it is linked to mathematical content such as numeration and measurement. In the curricular conceptions analyzed, geometric content is related to life, due to the repercussions that its study has in various areas and subjects of social activity.

Based on these ideas, the study of geometry begins with contents of spatial orientation, starting with playful activities and later with figures, movements and geometric bodies; however, in some countries, the study of geometric begins with geometric bodies and later introduces plane figures.

The teaching is oriented to the development of geometric skills: visual, verbal, drawing, logic and modeling; the problem-solving method is also used, explicitly or implicitly, for its development in schoolchildren, through activities of manipulation, creation, generalization and application of the contents.

Geometric content is taught intuitively to ensure spatial imagination and the formation of basic concepts with the use of concrete objects, such as models, so that the student discovers the properties of geometric objects and develops his or her vocabulary.

Finally, technological tools, such as videos and computer applications, are used in both European and Latin American countries. Van Hiele's model is applied to guide geometric understanding in schoolchildren, and there is the influence of national and international research on the methodological treatment of geometry in these grades.

The trends analyzed highlight, as guiding ideas for the direction of the teaching-learning process of Geometry in Elementary Education, the importance of linking geometric contents with life; the use of concrete objects and other teaching means to develop geometric skills; and the development of abstract geometric thinking.

1.3. The Teaching-Learning Process of Geometry in the Cuban Elementary School

The geometry process in Cuba has gone through deep transformations in its design and curricular development; it has been influenced by the perspectives of Mathematics Education and the different trends that have emerged at the international level.

The historical development of the teaching-learning process of geometry in the Cuban school;, according to Rizo Cabrera (1987), can be summarized in three stages: a traditional one, determined by the use of Euclidean ideas, which concludes in the 19th century; another one influenced by the axiomatic restructuring carried out by Hilbert to Euclid's postulates and Klein's proposals, which reaches the 1950s; and the last one, from the second half of the 20th century until the defense of his thesis, characterized by two important moments: the introduction of the theory of conjuncts in the mathematical curriculum and the ideas of the International Commission for the Study and Improvement of Mathematics Education (CIEAEM) to put an end to the teaching of pure mathematics and bring it closer to other sciences, in accordance with social reality and practice.

With Rizo Cabrera's doctoral thesis dissertation (1987), a new stage begins in the conception of the teaching-learning process of geometry in the Cuban school, in which geometric contents are structured in three fundamental moments: an initial or propaedeutic one, which covers preschool and up to the fourth grade; another of deductive study, which begins in the fifth and sixth grades of elementary school and extends up to Basic Secondary School; and the last one of complementation. It should be noted that at none of these times is there a rigorous axiomatic construction of geometry, although intuitive elements of the axioms that appear in Euclid's and Hilbert's systems are included.

In the first cycle of Cuban elementary school, the teaching of geometry has an intuitive, operative, perceptual and practical character, since all the properties of elementary geometric figures and bodies are obtained from visual and tactile perceptions. This character is given by the psychological particularities of schoolchildren in these grades, which have been studied by Cuban specialists such as Rico Montero et al. (2008), to provide better pedagogical attention and direct educational actions more effectively in the teaching-learning process.

Thus, the first cycle of Elementary Education is subdivided, due to the variety of ages, into two moments or stages of development: one from 6 to 7 years old (first and second grades) and the other from 8 to 10 years old (third and fourth grades).

According to the mathematics programs, in the first grade the formation of concepts is done with concrete objects or their materialization. In these grades, mental operations such as analysis, synthesis, abstraction and generalization are developed, with actions that favor the formation of notions and skills in schoolchildren, such as observation, description, comparison, classification, among others.

The third and fourth grades are the moment of development in which the first cycle culminates. In these grades, the forms of organization and direction of a reflective learning activity continue, based on the requirements indicated for the initial grades. It is possible to achieve, at the end of the cycle, higher levels in the development of skills in the students and an evaluative control of the activity carried out.

In the second cycle of elementary school, the teaching of geometry has, among its purposes, to continue the development of the skills and abilities initiated in previous grades. The fifth grade is the stage of the transition from inductive to deductive geometry. In this and the following grade, geometric knowledge and skills already obtained are systematized as preconditions for the development of new skills such as argumentation, supported by calculus and geometric demonstrations.

The teaching of geometry in elementary education should contribute to the development of schoolchildren's capacity for spatial representation and imagination (geometric view), through perceptual activities, of the shape and size of objects; to master the essential properties of geometric objects; and to develop skills in recognizing them, tracing and/or constructing them, describing them and arguing given geometric propositions.

The fundamental geometric object concepts of figures, bodies and geometric motions or transformations in the plane, shown in the table below, are addressed in Elementary Educaof dition.

Table 1. Fundamental Geometric Concepts Addressed in Primary Education.

Geometric figures		Geometric movements	Geometric bodies
Elementary geometric figures	**Other geometric shapes**		
Point Straight line Plane Segment Semi-line Semiplane Angle	Polygon: * Triangle * Quadrilateral: - Trapezoid - Parallelogram - Rectangle - Square - Rhombus Cirle	Symmetry Translation Reflection Rotation	Prism: * Orthohedron: - Cube Pyramid Cylinder Cone Sphere

It should be noted that in these grades there are concepts that are initiated and others, which serve as a basis for new concepts, continue to deepen. Ballester Pedroso et al. (2001), point out that the formation of concepts is of great importance in the teaching-learning process of Mathematics, since it contributes to the understanding of mathematical relations; the development of the capacity to apply what has been learned, in a safe and creative way; the logical-verbal training; the transmission of important ideological notions, related to the theory of knowledge; and the development of numerous properties of character.

For the methodological work on the formation of concepts, three stages or phases are considered, in which the schoolchild performs different actions; these are: identifying the concept, realizing it and applying it. The essence of the first stage lies in presenting a series of objects for the students to determine whether they comply with the properties of the concept. In the second stage, these properties are checked on the objects, and end with the explanation of the concept; while the third stage is carried out in relation to other teaching situations, which serve as preconditions for defining new concepts.

Another element to take into account in obtaining concepts is the way they are formed. For this purpose, we follow the inductive one, which goes from the particular (content) to the general (extension), because taking examples as bases, the definition is elaborated step by step; and the deductive one, which goes from the general to the particular (from the extension to the content of the concept), because we start from the definition of the concept and its content is discovered, by means of examples.

The fundamental relations that are worked on;, linked to the concepts of objects, are those of position between points, straight lines and the plane ("*...passes through..*". or "*...is located in..*".); between points that are located on a straight line ("*...is located between... and..*".); between straight lines ("... *is located between... and...* "); between straight lines ("*...intersect*" or "*...do not intersect"*), to indicate parallelism, in the case of those that do not intersect in the plane, or perpendicularity, those that when intersecting coincide with the short sides of the bevel; and congruence or geometric equality ("*...the same as..*".), between segments, which is later used, through superposition activities, in other figures.

This is achieved by performing operations such as recognizing geometric figures and bodies in environmental objects, tracing geometric figures, comparing segments, measuring the length of a segment, developing geometric bodies, and superimposing figures, among others.

The treatment of the concepts of relation and operation, together with those of objects, in the teaching-learning process of geometry, as Proenza Garrido (2002), points out, *"are transformed into procedures"* (p. 59). Thus, geometric procedures are those procedures typical of the treatment of geometry that are related to those typical of the teaching-learning process of mathematics: algorithmic and heuristic.

A geometric procedure is algorithmic when it offers a sequence of finite steps that allow the solution of the given problem situation, from which Sequences of Indications with

Algorithmic Character (SIAC) are obtained; for example, geometric constructions: elementary[1], fundamental and formal.

While geometric heuristic procedures are those oriented towards the search and discovery of solution paths based on analogy, induction, the reduction of one problem to another already solved, generalization, etc. Examples of situations where this type of geometric procedure is applied are found in some exercises of recognition of compound figures and argumentation of propositions, in which a relationship between concepts is established.

The Elementary Education stage is of great importance in the teaching-learning process of geometry, since it is at this stage that the ability to internalize the observed geometric properties and the formation of their geometric vocabulary begins to develop in schoolchildren.

The teaching of geometric concepts and procedures in Cuban elementary school has as background the intuitive work carried out in the program "*Elementary Notions of Mathematics*", which is included in children's circles, non-institutional tracks and the preschool grade; in addition to the contents of orientation in space and on the tracing sheet, which begins in the first grade.

The mastery of these concepts and procedures is achieved in schoolchildren through experimental activities of drawing, modeling, manipulation, superposition, composition and decomposition, in which a set of geometric skills are gradually developed.

[1]In plane geometry, elementary geometric constructions are those constructions (point and line), which together with the fundamental ones (parallel and perpendicular lines; angles; etc.), serve as the basis for formal geometric constructions (circumference, parallelogram, rectangle, etc.), where these geometric notions are applied.

02.

THEORETICAL FOUNDATIONS FOR THE DEVELOPMENT OF GEOMETRIC SKILLS IN ELEMENTARY EDUCATION

"From the ideas of space to the space of ideas".

(Calvo Penadés et al., 2002)

2.1. The development of geometric skills in Elementary Education

One of the fundamental objectives of geometry teaching in Elementary Education is the development of skills; a psychological category that allows an individual to execute a certain activity successfully, depending on the knowledge achieved; to know and interpret, as stated in the requirements of the elementary school model in Cuba (Rico Montero et al., 2008), the components of nature, the relationships that exist between them, society and oneself.

Mathematical skills, for Krutetskii (1968), cited by Wielewski (2005), are those "*individual psychological characteristics (mainly of mental activity) that respond to the demands of school mathematical activity and that influence.... successfully in the creative mastery of mathematics as a school subject*". (p. 32)

For their part, Geissler et al. (1977) state that they are "*those automated components that arise in the development of actions with a preferably mathematical content and finally contribute decisively, through their application, to the level of power in mathematics*". (p. 75)

Mathematical skills are developed only in activities related to mathematics. According to Wielewski (2005), this means that each of the components of this subject can contribute to the development of related and/or different skills. Those studied by Krutetskii (1968) are: perception, generalization, logical-ratiocination, reduction, flexibility, analytical-synthetic, mathematical memory and spatial concepts. Among these mathematical skills are implicit the basic geometric skills proposed by Hoffer (1990), cited by Galindo (1996), who covers them in five areas: visual, verbal, drawing, logic and modeling.

The development of visual skills enables schoolchildren to assimilate geometric properties from what they observe, whether real objects or representations; verbal skills enable them to use geometric vocabulary appropriately; verbal skills enable them to interpret and represent; logical skills enable them to acquire the necessary arguments to recognize the validity of a geometric proposition; and modeling skills enable them to describe and explain real-life phenomena by means of models.

From the analysis of these precedents, the term geometric skills is defined as: "*type of mathematical skills that enable an individual, from the domain of practical and intellectual actions and operations, to apply the geometric concepts and procedures acquired in the creative solution of situations specific to the subject and/or practical life*".

On the other hand, the study of Cuban elementary school mathematics programs (Cuba. Ministry of Education, 2007), and other materials of necessary consultation (Rizo Cabrera, 1987; León Roldán, 2007), allowed us to assess the development process of geometric skills in elementary education. These skills are: recognizing geometric objects, tracing and/or constructing, arguing geometric propositions and solving geometric problems, which are closely related to those pointed out by Hoffer (1990), cited by Galindo (1996), and those analyzed by Krutetskii (1968), mentioned by Wielewski (2005); and Giorgion (2010); although many of the latter classification, constitute actions for their development.

The development of the ability to recognize geometric objects in Elementary Education

The ability to recognize, which is developed from the first grades of elementary education, is considered basic, since it is an indispensable action (Ortiz Ocaña, 2006) for the development of other skills. In elementary education, geometric learning is directed in such a way that schoolchildren can recognize geometric objects in the environment, from models, through the denomination of a concept and in compound figures.

From the psychological point of view, in order to recognize, there must be a familiarization between the subject and the object, based on visual and/or tactile perceptions. *"When this object appears in our consciousness, a correspondence is established that produces the knowledge that this object had already been consciously recognized before".* (Ponce Solozábal, 1988, p. 61)

In fact, recognition is the last of the levels of perceptual actions together with discovery, differentiation and identification (Petrovski, 1986). The first two are listed among the perceptual actions and identification together with recognition among the recognition actions.

Perception is understood as *"the image of objects or phenomena that is created in the consciousness of the individual by acting directly on the sense organs, a process during which the ordering and association of the various sensations into integral images of things and facts takes place".* (Petrovski, 1986, p. 223)

Alsina Catalá et al. (1989), mean that the correct development of visual perception is fundamental to achieving the knowledge of spatial relationships, because through it the subject can analyze the shape, size and distribution of objects in the environment, with respect to the location of his own body.

In this regard, the literature consulted (Alsina Catalá et al., 1989) cites three types of space where these and other skills can be developed: micro space, a reduced area where the schoolchild can carry out experimental activities (table); meso space, a part that is within sight, where small movements can be made and in which fixed objects function as points of reference (classroom, playground); and macro space, an area framed in the open air (city, countryside).

The development of visual perception requires *"a series of skills, among which 'knowing how to see' and 'knowing how to interpret' stand out"* (Alsina Catalá et al., 1989, p. 61); these actions are considered to be related to the levels of perceptual actions addressed by Petrovski (1986). For this author, discovery constitutes the initial basis of all sensory processes, while differentiation has as its final result the formation of the perceptual image.

On the other hand, recognition can take place (Petrovski, 1986) once the image of perception is formed. For this, the operations of comparison and identification are indispensable. The latter is the intermediate part between differentiation and recognition, which always involves identification, where the perceived stimulus or image is recorded in memory. Recognition also includes the classification into classes of objects, the choice of the corresponding model among them and the comparison, based on the essential properties of the object, stored in long-term memory.

It is assumed that although recognizing, according to Silvestre Oramas & Zilberstein Toruncha (2002), is not indicated as an intellectual ability, it can be considered as such; since, according to Petrovski (1986), recognizing always involves identifying, which is considered, by these authors, an intellectual ability.

Intellectual skills, according to Álvarez de Zayas (1999); Silvestre Oramas & Zilberstein Toruncha (2002); and Ortiz Ocaña (2006), are skills that are generally applied both in the teaching-learning process and in life, and developed from specific skills. They are a precondition for the development of other skills and for the rest of man's cognitive activity. One element to take into account for the development of the ability to recognize geometric objects is the algorithm proposed by Talízina (1988), which consists of the following steps: (1) denominate the first characteristic; (2) establish whether the object has the first characteristic; (3) note the result obtained; and (4) check whether the answer is correct.

According to this author, in the first step, one of the characteristics of a geometric object is expressed; in the second step, it is determined whether the object complies with that given characteristic; in the third step, the result is noted down, and it is continued by analyzing whether the object continues to comply with the rest of the characteristics; and in the fourth step, the conclusion is reached whether it complies with all of those stated.

It is considered that this procedure, which is for the formation of concepts, in a general way, can be applied to the development of the ability to recognize geometric objects. However, it does not state explicitly the psychic phenomena involved in the development of skills such as visual and tactile perception, necessary for the teaching of geometry in these grades. In this proposal of Talízina (1988), it is not proposed as a step to establish links between objects, which would allow establishing the differences and similarities that exist between them.

The development of geometric tracing and/or building skills in elementary education.

The geometric skill of tracing and/or constructing has important functions in the geometry class, since its development allows schoolchildren to obtain geometric figures and bodies as representatives of any concept of the subject, to understand their properties or from their mastery: on graph paper, with drawing instruments or other materials such as the tangram, the geoplane, rods and flat developments.

In elementary school, students become familiar with drawing instruments such as the drawing template, to represent circles and other polygons; the ruler, to draw straight lines and other figures; the bevel, which allows in combination with a ruler or a second bevel to draw parallel and perpendicular lines; and the compass, to draw circles.

The first grades form the basis for the formal geometric constructions that begin to develop in grade four. In general, geometric constructions are accompanied by a procedure that is initially exemplified by the teacher on the blackboard and later assimilated by the students through exercise.

Geissler et al. (1978), present a series of steps that the teacher has to bear in mind to develop this skill in their students, which are related to the theory of the stage formation of mental actions and concepts, since these authors applied the ideas of Galperin (1987), in the formation of all mathematical concepts, from grade one to grade four, but they correspond only to the stage of the action of material or materialized form, of the execution phase. These methodological steps are summarized below:

1. Demonstrate on the blackboard, or by using another type of teaching aid, how the construction is carried out.
2. Carry out the construction on the blackboard based on the steps presented by a student. At this stage, the student begins to appropriate the geometric vocabulary necessary to carry out the construction; gradually it is perfected.
3. Describe the procedure, while a schoolboy follows the steps of the description. In this step, a demonstration on the blackboard by the schoolchildren is not advisable since the instruments for tracing on the blackboard are difficult to manipulate (although from fourth grade onwards they are already in a position to manipulate them).
4. Provide textually the procedure to follow to carry out the construction; the students read it and immediately follow the instructions to carry out the construction.

In Elementary Education, in order to develop the geometric skill of tracing and/or constructing, schoolchildren reproduce geometric objects with different materials and instruments, through the observation of models; they represent them following a set of steps (SICA), their essential properties and the relationships between concepts; finally, they are able to obtain other geometric objects, through composition and decomposition activities.

The development of the ability to argue geometric propositions in Elementary Education

Arguing is an intellectual skill (Silvestre Oramas & Zilberstein Toruncha, 2002). It consists of a reasoning process in which a position is defended based on a group of reasons that support it. Argumentation influences the development of general skills such as refutation and inference, since during this process the acquired knowledge is mentally related in the form of ideas, which is then materialized in the form of criteria and/or judgments, verbal or written.

Ponce Solozábal (1988) states that ideas *"are abstract and indirect products of cognition; they are new knowledge obtained from previous knowledge"* (p. 36). The psychologist Vygotsky (1982), emphasizes the importance between thought and language; he states that *"external language is the conversion of thought into words, its materialization and objectification"* (p. 129), while in inner language, speech is transformed into inner thoughts. According to this same author, *"written language is the most elaborated form of language"* (Vygotsky, 1982, p. 140), since it facilitates a better organization of the ideas of what is transmitted and can be revised on multiple occasions.

The argument or judgment, as Petrovski (1986) names it, *"is the reflection of the existing connections between objects and phenomena of reality or between their properties and characteristics"* (p. 297). This author considers that judgments are manifestations of something about something that affirm or deny relations between objects, events and phenomena of reality.

Judgments are usually formed by two methods, and their purpose is the formation of conclusions. By the direct method, what is perceived is expressed, and from conclusions and reasoning, by means of a set of judgments, other judgments are founded.

In the teaching of geometry in elementary education, the development of the ability to argue geometric propositions favors the formation of geometric vocabulary, logical-deductive thinking and lays the foundations for geometric demonstrations, which are worked on in later grades.

Hoffer (1990), who is mentioned by Galindo (1996), agrees with Van Hiele (1957), in that when developing this skill, in their first signs, schoolchildren, since they do not master the properties of geometric figures and bodies, interpret oral or written phrases that describe them. Later, they are able to describe geometric figures and bodies and use their properties to argue the truth value of geometric propositions based on the known ones and the relationship between geometric concepts.

For the ability to argue in geometry, two essential ways stand out, one on the experimental basis and the other on the basis of properties, although sometimes they are combined.

1. On experimental basis.
2. On the basis of properties.

- Identify or realize concepts or relationships.
- Apply a previously known proposition.
- Apply a procedure.
- Apply the counterexample of a theorem.
- Refute a proposition by means of a counterexample.

Demonstrations constitute a higher level of the argumentation skill, and although the objective is that students understand and become familiar with them, it is necessary that teachers have a mastery of them in terms of the geometric content that is shown in them, as well as their methodological treatment, for which reason some elements of the process of elaboration and demonstrations of theorems are presented below.

In the treatment of theorems and their proofs, the following partial processes are highlighted: search for the theorem, search for a proof, and representation of the proof.

The search for theorems constitutes a partial process that has special significance from the pedagogical point of view, since it makes it possible to place schoolchildren before situations in which they act as "*re-discoverers*".

In the case of geometry teaching in elementary school, the treatment of theorems begins in sixth grade. The search for theorems is more important than their proof, although in the remaining grades properties that constitute theorems are dealt with, but due to methodological factors they are not treated in this way.

For example, in the first cycle of elementary school, it is a goal for students to recognize that the opposite sides of a parallelogram are congruent, a proposition that constitutes a theorem; however, it is not proved. Of the reductive procedures, practical activities such as measurements and systematic comparisons are the most used in elementary school.

The development of geometric problem-solving ability

The development of problem-solving skills is an elementary objective of the General Education Mathematics program; therefore, teachers should pay special attention to its methodological treatment.

We consider it important to address some theoretical considerations on the problem concept. From the scientific point of view, it is convenient to specify the different connotations of this concept. As a category of dialectical logic, it reflects the existence of a dialectical contradiction in the object to be known, since it "*determines the research activity of man's search, aimed at the discovery of a new knowledge or the application of a known one to a new situation*". (Majmutov, 1983, p.58)

As a psychological category, it reflects the contradictions within the process of knowledge of the object by the subject. According to Rubinstein (1966) it is the elementary cause of thinking since "*the process of thinking starts from a problematic situation*" (p. 109). He establishes a difference between the problem situation and the problem itself; the former is the one that presents unknown, unclear or explicit elements, while in the latter the subject is aware of what is sought.

For his part, the psychologist Labarrere (1987), states that "*the psychic activity of the subject*" is involved in every problem. (p.6)

As a mathematical concept, different authors have given their criteria. Polya (1976) maintains that "*a problem means* *consciously searching with some appropriate action to achieve a goal clearly conceived but not immediately attainable"* (p. 11); Fridman (1977), is of the criterion that it is *"a model of the problem situation, expressed with the help of the symbols of any natural or artificial language".* (p.15)

For their part, Campistrous & Rizo (1996), state that it is *"any situation in which there is an initial approach and a requirement that forces to transform it. The way to go from the initial situation or approach to the new required situation must be unknown, and the person must want to make the transformation".* (pp. 9-10)

In summary, the problem concept is characterized by the aspects:

1. It expresses an initial situation (initial conditions or data) and a final one (requirements).
2. The path must be unknown.
3. The subject feels the need to solve the problem.
4. It demands intense cognitive activity.

In particular, geometric problems are not only those in which students follow a certain formula and/or procedure to find the solution. In general, geometric problems are considered to be those situations where the arithmetic component moves to a lower level and where the mastery of geometric properties becomes more important for their solution.

There are different classification criteria, for example:

I. According to the way in which the information contained in a problem is written, the correlation of the known (conditions or data) and the unknown (demands), as well as the type of mental activity performed by the student for its solution. Cruz Ramírez (2006), citing Pehkonen (1955), proposes a classification, which we will now contextualize in the case of geometry:

CLOSED GEOMETRIC PROBLEMS

- Algorithmic geometric problems.
- Heuristic geometric problems.

OPEN GEOMETRIC PROBLEMS

Geometric problems are closed when they contain in their structure all the necessary information for their solution, which allows the solver to easily find the solution. This type of problem is subdivided according to the procedures used by the students in algorithmic and heuristic.

II. Algorithmic geometric problems are considered to be those whose solution follows, in an orderly manner, a succession of steps or procedures.On the other hand, geometric problems can be classified according to the nature of the requirement into problems of geometric construction, problems of demonstration or problems of geometric calculation.

1. Geometric construction problems are those in which the requirement is to obtain a figure that must satisfy certain properties or relations between elements or magnitudes.
2. Demonstration problems are those in which a geometric relation or property is demonstrated. This includes problems of demonstration of theorems.
3. Geometric calculation problems: those in which the requirement is to determine numerical values.

III. In addition, in all the literature on mathematics teaching methodology, there is a consensus in classifying problems according to the number of steps that have to be performed. Although there is a tendency to consider only this classification when referring to arithmetic problems, it is convenient to clarify that this classification is valid for geometric problems.

1. Simple problems: in cases that are solved with only one step.
2. Compound problems: in those solved with more than one step, compounds are subdivided into independent compounds and dependent compounds.

IV. Another classification criterion is related to the possibility or not of solving the problem. If the problem can be solved, then it is said to be solvable and otherwise not solvable. It should also be taken into account whether the initial conditions express the information explicitly or implicitly.

In the case at hand, geometric problems, in addition to the relations or values of the quantities of magnitudes, the representation of the figure in question may or may not appear.

The mastery of all these classifications allows teachers to know the different characteristics of conditions and exigencies, as well as the relationships between concepts and their properties, which will allow teachers to select in a graded manner the geometric problems they will propose to their students.

What has been analyzed so far allows us to state that in the solution of geometric problems it is not enough for the student to know only formulas, since knowledge of the properties of figures and bodies, the construction or graphic representation of the data, the recognition of equivalence relationships and the conversion of units of length, area and volume play an important role.

2.2. Principles for the teaching-learning process of geometry in Elementary Education

Rosental & Ludin's philosophical dictionary (1981), states that the term principle comes from the Latin "*principium*" and means "*starting point, guiding idea, fundamental rule of conduct*". In other words, principles are bases, foundations and postulates on which a science or discipline is based. Accordingly, each science establishes its own principles and theories to provide solutions to problems in its field of action.

Didactics of Mathematics as a scientific discipline, in constant changes and transformations, also proposes its own principles by which it is governed and which constitute contributions to general didactics. We share the criterion that "*the methodological treatment of each teaching content has its peculiarities and determines points of view or general postulates for its treatment*". (Barcia Martínez, 2000, p. 40)

For this reason, the most general postulates of the teaching-learning process of geometry in elementary education are determined. These principles are closely related to the didactic principles exposed by Labarrere Reyes & Valdivia Pairol (1988), since their more general aspects have been taken into account in the process of this educational subsystem.

The principles that govern the teaching-learning process of geometry in Elementary Education, and that affect the development of geometric skills were structured taking into account their meaning, foundation and actions for their application in the teaching-learning process. In the following sections, the five principles for the development of geometric skills in elementary education are presented.

1. Principle of continuous support in historical knowledge

This principle means that, in the teaching-learning process of geometry in Elementary Education, the contents of the history of geometry should be imparted, in correspondence with the advances of this science and the specificities of each grade. This is based on the fact that the teaching of geometry is the product of knowledge that has been developed and perfected over time. On the other hand, familiarizing schoolchildren with elements of the history of science (Savin, 1972), accessible to them, contributes to the scientific conception of the world.

According to De Guzmán (1993), *"the historical perspective brings us closer to mathematics as a human science, not deified, sometimes painfully crawling and sometimes fallible, but also capable of correcting its mistakes"* (p. 6). This same author, in highlighting the use of history in the teaching-learning process of the different subjects of the curriculum, states that "*it would be extraordinarily convenient that the different subjects we teach benefit from the historical vision... and that all our students be provided with even a brief global panorama of the historical development of the science that will occupy them all their lives*". (De Guzmán, 1993, p. 7)

The following are enunciated as actions for the application of this principle: (1) to elaborate historical curiosities in accordance with the geometric contents that are addressed in the cycle and the particularities of the students; (2) to orient activities that are related to practical problems that have been presented to man traditionally; (3) to emphasize the geometric knowledge developed by man throughout the history of mankind; (4) to elaborate teaching means that favor the use of historical curiosities.

Thus, for example, when schoolchildren begin to use the first spatial relationships to orient themselves in three-dimensional space (left-right, front-back and up-down), they can learn that the notion of distance was one of the first geometric concepts discovered by man and that the estimation of the time needed to make a trip led him to use the straight line, which is the shortest distance between two points. It should also be noted that in the history of mankind, man has oriented himself in the environment from the observation of the stars and has used different instruments, which, as is to be expected, have been perfected over time.

The study of geometric figures can be the moment when schoolchildren learn that the circle is the geometric figure that man first knew; this is because in the observation of the environment that surrounded him, he associated it with the sun and the moon. They can also learn that the knowledge that ancient cultures had about the circumference gave man the possibility to create the wheel and with it the cart, but also the properties of the circumference were used by man to measure time.

When using measuring and tracing instruments, the teacher can use the emergence and development of geometry as a science to motivate his class, letting his students know that the Egyptians were the first to start measuring, since they needed to constantly measure the borders of their lands, which were erased by the floods of the Nile River; also, the ruler and the compass have been the most used drawing instruments to solve geometric construction problems, since the Greek geometricians.

In the study of geometric bodies, schoolchildren can learn that dice (cube-shaped), according to findings, were used by the ancient Egyptians to play; for a long time, man did not know the spherical shape of the earth, so he represented it in different ways.

In the second cycle, when geometric problems are solved, schoolchildren can learn that many aspects of geometry, as mentioned above, respond to the need to solve problems not only in agriculture and architecture, but also in various social spheres.

On the other hand, by teaching the pyramid, the teacher can highlight the geometric knowledge of the Egyptians and the pre-Hispanic inhabitants of Central and South America, who were builders of great pyramids.

In fact, the development of teaching aids by schools can also lead teachers to use the history of geometry. Examples of these teaching aids can be models of geometric figures and bodies and the tangram, which is a game that originated in ancient Chinese culture (León González & Barcia Martínez, 2010).

It is considered necessary for teachers to use historical knowledge as an instrument during the introduction of geometric content, based on the use of curiosities according to the content in question. To this end, they must have a mathematical culture to deploy a developmental teaching-learning process in accordance with the biological and psychological characteristics of schoolchildren, as long as it is admitted in the process.

This principle is fundamentally related to the principles of the scientific nature of teaching and the principle of accessibility, proposed by Labarrere Reyes & Valdivia Pairol (1988), which express the need to include in the teaching-learning process the results of the development of science; the teacher plays a decisive role in relating the new knowledge with the thinking mechanisms of schoolchildren, in accordance with their age.

According to what has been analyzed above, the fundamental idea of this principle lies in the understanding, on the part of schoolchildren, that the historical development of geometry is linked to man's social progress and his desire to transform the world for his benefit.

2. Principle of the contextual character of geometric contents

This principle means that the teaching-learning process of geometry should be linked to real situations and the environment so that students can contextualize the geometric knowledge and skills acquired in the solution of practical situations and argue what they have done, depending on their particularities. It is based on the fact that the most ancient concepts of classical geometry arose as a result of the interaction of men with nature, who came to the knowledge of geometric shapes from the observation of the environment. On the other hand, it constitutes one of the reasons to which pedagogues of different times have referred and for which it is currently advocated in the teaching-learning process of geometry.

In this regard, Canals Tolosa (1997) points out: "*Geometry must be lived at school and throughout life; it must be, both for us and for our students, an opportunity to increase our capacity for discovery, our initiative and creativity, and our sensitivity to the beauty of forms, appreciated both in art and in nature and in the globality of the environment that surrounds us. It is necessary that together we learn to look at our environment with more geometric eyes and that both in the street and in the classroom we are happier doing geometry*". (p. 31)

This author adds: "*This idea of Geometry learned intuitively from everyday life and reinforced in some aspects by appropriate school practices, will be in the following lines as a starting point and I would also like it to be a backdrop that reappears from time to time*". (Canals Tolosa, 1997, p. 31)

Similarly, he points out that for the introduction and better fixation of geometric concepts, the teacher should not only show his students models and illustrations of geometric figures and bodies, but should also help them to observe the real properties of those figures and bodies present in their immediate environment, thus contributing to a better construction of geometric knowledge.

This point of view is shared by Alsina Catalá et al. (1989), who state that "*the environment, in its broadest sense, has been and will continue to be the great challenge, the great source and source of geometric studies, not only to motivate descriptions and models but, most interestingly, so that these geometric results can have an impact on the transformation of reality*". (p. 28)

For his part, Petrovski (1986), considers that, in these spatial activities, "*the shape, size, location and displacement of objects among themselves, as well as the simultaneous analysis of the location of one's own body with respect to the surrounding objects, are determined in the process* *of motor activity of the organism and constitute a specific higher expression of the analytical-synthesizing activity that has been called analysis of space*". (p. 245)

The literature consulted (Alsina Catalá et al., 1989) cites three types of space where geometric skills can be developed: (1) micro-space, reduced space where the child can perform experimental activities (table); (2) meso-space, space within sight, where small displacements can be made and in which fixed objects function as points of reference (classroom, playground); (3) macro-space, space of large dimensions, framed in the open air (city, countryside).

The actions proposed for the application of this principle in the teaching-learning process are the following: (1) develop spatial activities that allow the observation of geometric properties in the environment and abstractions; (2) emphasize the importance of geometric knowledge for life; (3) guide tasks that involve solving problems of practical life from geometric content.

In the formation of the first geometric notions in schoolchildren, the environment should be used as a starting point to favor the transition from concrete to abstract thinking and contribute to the development of geometric skills. Therefore, space exploration activities should predominate in which they recognize physical objects that are related to geometric shapes, reproduce them using geometric figures and bodies, and explain the reason why they have a certain shape.

Similarly, the treatment of geometric movements should begin with the perception of elements of the surrounding world, since it is from the linking of these contents with life that the process of abstraction and generalization of this knowledge takes place.

This principle is essentially related to the principles of the relationship between theory and practice and the conscious and active character of students under the guidance of the teacher, as stated by Labarrere Reyes & Valdivia Pairol (1988), since in order to achieve greater development of geometric skills, the teacher must structure practical activities in which students are involved with life situations.

It is important to emphasize the need for the teaching-learning process of geometry to always be related to situations in the social and cultural environment. This relationship helps schoolchildren recognize the presence of geometry in practice and to apply their geometric knowledge in different spheres.

3. Principle of the use of concrete objects and other teaching methods

This principle means that the use of real (concrete) objects and other teaching aids such as technological resources in the teaching-learning process of geometry in Elementary Education, under the guidance of the teacher, constitutes the fundamental basis for schoolchildren to appropriate, according to their difficulties, possibilities and interests, geometric concepts and procedures and use them in the solution of practical life problems; is based on the use of the inductive method (from the particular to the general) to reinforce direct observation and manipulation as essential methods for obtaining knowledge and the development of geometric skills.

In the teaching of geometry, the use of concrete objects as a teaching medium guarantees a three-dimensional approach to space and a direct link with nature. Concrete objects are those that have a real and material or physical existence, including geometric models[3]; but only in exceptional cases should graphic representations and illustrations be used as a substitute for them in the introduction of a geometric concept.

The use of concrete objects in the teaching-learning process of geometry contributes to the development of geometric skills, which affect the development of intellectual skills such as: observing, describing, comparing, classifying, defining, arguing, modeling, among others.

Thus, Piaget, quoted by Brihuega Nieto (2006), considers that schoolchildren "*the more time they devote to the study of the concrete, the more time they spend in observation, the better they will then move on to the understanding of abstract forms*". (p. 2)

Alsina Catalá et al. (1991), point out that "*geometric teaching should not succumb to the formal, symbolic or algebraic limitations of mathematical knowledge: it is precisely in this first stage of sensitivity that touch, sight, drawing and* *manipulation will familiarize the student with a whole world of shapes, figures and movements on which to base abstract models later on*". (p. 1)

Referring to the use of models in the teaching-learning process of geometry, Junquera Muné (1961), rightly states that "*the educator should not believe that intuition is enough in the sense of showing the bodies. He should not have them in view of the children, but put them in their hands so that they have them and retain them, observing them. There should be several series of bodies, either any or geometric, and within these series of each class, varying the size, the number of faces, etc. To insist on "doing" geometry with a cube, a prism, a pyramid, etc., is vanity*". (p. 512)

That is why in the teaching of geometry, the observation and manipulation of concrete objects and models from different points of view, by schoolchildren, constitutes the fundamental basis for acquiring, in a sensory-perceptual way (through sight and touch), the first properties of geometric concepts. According to Galperin (1987), in the acquisition of concepts, thinking moves from the concrete to the abstract, and from the abstract content of the concept to the broad knowledge related to it, in which the concept is applied to new particular cases.

Other teaching aids that should be taken advantage of for the development of geometric skills in schoolchildren are the different technological resources (audiovisual media and computer applications), which, when used in the educational context, perform a series of basic functions, typical of these media, apart from the specific ones, determined by the use that teachers make of them. The American Council of Teachers of Mathematics (2000), emphasizes the importance of technology in the mathematics taught since it is considered an essential tool for teaching, learning and "*doing*" mathematics.

The actions proposed for the application of this principle are as follows: (1) use concrete objects in the introduction of geometric concepts and procedures; (2) guide experimental activities for schoolchildren to discover geometric properties; (3) encourage the elaboration of geometric models by schoolchildren; (4) select, evaluate and use technological resources in the teaching-learning process of geometry.

Thus, when elaborating the cube concept with the help of a superior generic concept (in this case orthohedron), through the indication of one or several formative characteristics of types (it has square and equal faces), schoolchildren are presented with different examples of the same class and other classes to analyze them, compare them and identify in them the common essential characteristics (invariables), which after being found are abstracted and synthesized mentally.

It is then, when the teacher introduces the name of the concept and begins to elaborate the definition, step by step, with the help of the students and depending on each of the grades; later, the elaborated concept is deepened through the search for new representatives.

From this point of view, it is considered that once schoolchildren have manipulated the object, they will be able to mentally generate the known properties and transfer or adapt them to the solution of analogous and/or new situations.

On the other hand, it is useful for the development of geometric skills in elementary education that students are able to make, on their own, with the help of the teacher or other classmates, models of geometric figures and bodies, with different materials such as paper, cardboard, plasticine, wire, wood, lace, scissors, colors or tempera.

It is also considered important to carry out experimental activities on grids and on the geoplane so that schoolchildren understand the properties of geometric figures and movements and can apply the knowledge acquired in the resolution of geometric problems—the foundation of the truth value of propositions and constructions.

In relation to the use of educational technology in the teaching-learning process of geometry in elementary education, it should be noted that, sometimes, it has disadvantages with respect to the rest of the traditional means of teaching this subject, especially in these grades in which the development of geometric skills is carried out from real situations in the environment. However, it cannot be forgotten that they provide teachers, to a great extent, the opportunity to channel geometric contents in a more attractive and interesting way.

That is why it is essential for teachers to take advantage of the benefits that the technological process puts at their disposal, without ever undermining the value of traditional means, to offer schoolchildren knowledge in a more novel way and in accordance with modern times, since their use in the teaching-learning process of geometry is necessary due to the demands that society as a whole exerts on the school and for the achievement of developmental learning.

Although the unity of instruction and education is present in all the proposed principles, that of the use of concrete objects and other means of teaching is mainly linked to that of the educational character of teaching, in addition to the principle of the audiovisual character of teaching: union of the concrete and the abstract, that of attention to individual differences within the collective character of the teaching-educational process and that of the conscious and active character of the students under the guidance of the teacher, raised by Labarrere Reyes & Valdivia Pairol (1988), since it establishes correspondence between the objectives of education, the laws of physical and intellectual development of schoolchildren, the scientific-technical revolution and the demands of the contemporary elementary school.

4. Principle of intra- and inter-subject relationships in geometry teaching

This principle means that the teaching-learning process of geometry should be related to other parts of mathematics and subjects of the curriculum. It is based on the need to achieve solidity in the assimilation of geometric knowledge and skills.

The actions proposed for the application of this principle are the following: (1) to relate the teaching-learning process of geometry with the components: numeration, calculus, magnitudes, arithmetic problems, statistics, and other parts of mathematics; (2) to link the teaching-learning process of geometry with the components of the subject Spanish Language and the subjects: The world we live in (first cycle), Natural Sciences (second cycle) and Plastic Education.

In the teaching-learning process of geometry in Elementary Education, the contents of other parts of Mathematics serve as an aid in the acquisition of geometric skills. Such is the case of the components: numeration, calculus, magnitudes, arithmetic problems and statistics.

In the numeration component, in the early grades, the sets formed by geometric figures and bodies serve as a basis for identifying, representing, reading and comparing natural numbers. They are also used to solve formal exercises of the four fundamental calculation operations: addition, subtraction, multiplication and division. Similarly, from third grade on, geometric figures are often used for the graphical representation of fractions.

On the other hand, there is the magnitudes component, which has been linked to this branch of mathematics since the emergence of geometry, since throughout history many geometric constructions have been made with the help of units of measurement.

By linking the contents of the magnitudes and geometric components, schoolchildren can estimate and measure lengths, as well as carry out formal geometric constructions and solve geometric problems, since the performance of activities of comparison of quantities of figures and geometric bodies, together with the introduction of the concept of consecutive sides, are preconditions for the treatment of the contents of perimeter, area and volume.

The part-whole relationship, which is widely used in solving arithmetic problems, is a way to develop the ability to recognize geometric objects. From this relationship, it is possible to isolate, from its context, the general figure, which is composed of several parts. It is also considered appropriate to use elements of combinatorial theory (León González & Barcia Martínez, 2006) as long as the situation allows it in order to establish strategies that contribute to developing this skill in schoolchildren.

In Elementary Education, from the third grade onwards, the first notions of statistics are also taught in the resolution of arithmetic problems, linked to situations in different spheres of life, where tables and graphs that represent numerical data geometrically are interpreted by schoolchildren.

But geometric concepts and procedures can also be related or linked to the rest of the subjects in the curriculum so that students can understand their applicability in different contexts. The components of the subject Spanish Language: grammar, textual production, reading, calligraphy and spelling, help students in these grades to understand the properties of geometric figures, bodies and movements; to use geometric terms appropriately when naming these geometric objects; and to present fluently and coherently, either orally or in writing, their arguments about the veracity of any geometric proposition.

The potential of nature for the study of geometric situations can also be seen in the subject "*The world we live in*", which helps schoolchildren to observe and understand the natural and social processes occurring at a given historical moment. Its link with geometry classes can constitute one of the basic pillars for its teaching, since through its study, schoolchildren can perceive the diversity of shapes in the environment and the reason why certain "*living beings*" and "*non-living objects*" are related to a specific geometric shape; determine on a plane the real distance between two points by measuring segments; orient themselves in space, by means of the cardinal points (North, South, East, West). They can also explain the relationship of geometric movements with physical phenomena that occur around them, such as the succession of days and nights; the change of seasons, from the rotation and translation movements of the planet Earth; the projection of a movie, the movements of an elevator, among others.

Another of the curriculum subjects with which the teaching of geometry in elementary education can be linked is Plastic Education. It should be recalled that for a long time the curricula for the teaching of geometry contributed to the aesthetic formation of schoolchildren, based on artistic appreciation and creations.

Plastic Education facilitates the integral formation of schoolchildren, since it is related to all subjects of the curriculum and provides a high level of knowledge that schoolchildren reflect in their graphic representations. It is also aimed at "*familiarizing children with all the phenomena and objects of the surrounding world, both natural and man-made*". (Ruiz Espín et al., 2000, p. 30)

The activities of painting, drawing, modeling, cutting, composing and pasting facilitate the development of diverse mental actions and promote a greater development of skills, in addition to perfecting visual and spatial perception. Likewise, activities of sensory differentiation of shape, color, texture and size of natural, graphic and industrial objects are necessary for both subjects.

This principle is mainly related to that of solidity in the assimilation of knowledge, skills and habits and to that of systematization of teaching, as determined by Labarrere Reyes & Valdivia Pairol (1988). By linking geometric contents with other components of mathematics and other subjects in the curriculum, the teacher can help students consolidate and assimilate the knowledge and skills acquired, which will be evaluated by the teacher throughout the process of skill development.

5. Principle of abstract geometric thinking

This principle means that the teaching of geometry should start from the real experience of schoolchildren until theoretical generalizations are reached. It is based on the need to conveniently use the analytical, synthetic, inductive and deductive processes from an early age, in order to stimulate the intellectual development of schoolchildren and the applicability of geometric contents.

Geometric thinking, according to Proenza Garrido (2002), "*is a form of mathematical thinking, but not exclusive to it and is based on the knowledge of a model of three-dimensional physical space*" (p. 37). On the other hand, abstract thinking is the psychic process that allows an individual to understand concepts of objects and establish relationships between them, without having them in a concrete way.

Abstract thinking begins to develop, in school-age children, in its simplest forms and always, on the basis of practical and sensory experience, since it is never completely detached from sensations, perceptions and representations, until higher levels are reached in adolescence, when it begins to operate, not only with isolated concepts, but with classes or complete systems of concepts; since "*abstraction presupposes a mental division of the phenomenon or object* *into its properties, relations, parts, degrees of development, etc*". (Konstantinov et al., 1980, p. 240)

The following actions are proposed for the application of this principle: (1) gradually increase the levels of complexity of the proposed tasks until the representations of geometric figures and bodies are dispensed with, whenever possible; (2) develop activities in which the students establish relationships between concepts; (3) propose activities in which the students generalize the geometric properties known by sensory-perceptual means in the solution of problematic situations of the social and cultural environment.

As has been indicated, in elementary education, abstract geometric thinking begins to develop from the observation and manipulation of concrete objects, since through these activities the ability to analyze, compare and generalize is developed, and once the mastery of the properties of geometric figures, bodies and movements increases, and the ability to establish relationships between concepts is expanded.

It is curious to note how the experimental activities in which schoolchildren begin to recognize, understand and discover the properties of figures (number of sides and vertices, length of opposite and consecutive sides; the position relationship of opposite and consecutive sides) and geometric bodies (number of faces, vertices and edges, whether they are bounded by flat, curved or flat and curved surfaces, etc.), will later be used to recognize different (variable) and common (invariable) characteristics between figures and geometric bodies; the relationships between geometric concepts (mainly the superior and inferior order, according to grades); draw and/or construct geometric figures and bodies, where the relationship between concepts is established; and argue the veracity of given geometric propositions.

Likewise, activities where the students recognize the procedure used to move figures in the plane; in addition to performing some simple movements, it allows them to understand the properties of each of the geometric movements and the generalities that are established between them.

This principle reaches its maximum expression in that of the audiovisual character of teaching: union of the concrete and the abstract, that of the conscious and active character of the students under the guidance of the teacher and that of the attention to individual differences within the collective character of the teaching-educational process, stated Labarrere Reyes & Valdivia Pairol (1988), since abstract thinking, according to the Marxist-Leninist theory of knowledge (Lenin, 1964; Konstantinov et al., 1980) is not the last stage of the process of knowledge; the most important is the ascension of thought from the abstract to the concrete, when the scholar comes to represent and transform objective reality. It is in this way that the geometric concepts and procedures acquired through practice, with the teacher's help, are used again in practice when they provide solutions to everyday life situations.

Systemic nature of the proposed principles

The proposed principles are intimately linked and form a system, articulated according to the aims of education. Thus, each principle fulfills certain objectives in the teaching-learning process of geometry, and the purpose of one is subordinated, in an orderly manner, to the rest of the system.

A system is understood as "*a set of interrelated components, from a static and dynamic point of view, whose functioning is aimed at achieving certain objectives, which make it possible to solve a problematic situation, under certain external conditions*". (Álvarez de Zayas, 1989, p. 25)

Taking the above definition as a reference, it can be pointed out that the principles for the development of geometric skills in Elementary Education form a system due to: the relationships established among them; the influence they exert on the categories of the teaching-learning process; the common purpose they pursue (to contribute to the development of geometric skills in Elementary Education); and the hierarchization and centralization of the principle of the contextual character of geometric contents as a guiding element.

Thus, the principle of continuous support in historical knowledge is fundamentally related to the contextual character of geometric contents, to the use of concrete objects and other means of teaching, and to abstract geometric thinking. This relationship is given in that the historical advances of geometry are found in the solution of problems that have been presented to man in the social and cultural environment over the centuries and that have contributed to its development as a science.

In the same way, the study of concrete objects, through time, has enriched geometric knowledge; but in addition, the history of geometry can be used in the treatment of the teaching-learning process by elaborating teaching aids and developing skills. On the other hand, it can be affirmed that the solution of everyday problems, based on geometric abstractions, has historically served as a basis for the consolidation of the foundations and postulates of geometry.

The principle of using concrete objects and other teaching aids is linked, in addition to the principle of continuous support in historical knowledge, to that of the contextual nature of geometric contents and that of the intra- and inter-mathematical relationship of geometry teaching. This link is found in the fact that the concrete objects and teaching aids used for the treatment of geometric contents and the development of skills are related to reality; thus, a relationship is established between the known and the unknown, after the study of other components of mathematics and subjects of the curriculum.

The principle of intra- and inter-subject relationship in the teaching of geometry is related, in addition to the principle of continuous support in historical knowledge and the use of concrete objects and other teaching aids, to the principle of the contextual character of geometric contents and abstract geometric thinking, since in order to achieve solidity in the assimilation of knowledge and the development of geometric skills, it is necessary to start from the problematic situations presented to students in the social and cultural context in which they live in order to reach theoretical generalizations.

Finally, the principle of abstract geometric thought is directly linked to all those proposed, as a logical consequence of the Marxist-Leninist theory of knowledge, since it reflects the unity of the concrete and the abstract.

The following scheme illustrates the relationship between the proposed principles.

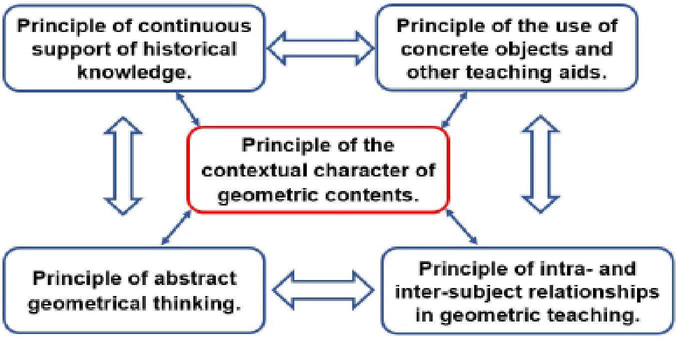

Figure 1. Systemic character of the principles for the development of geometric skills in elementary education.

The "*principle of the contextual character of geometric contents*" is considered as the central axis for the development of geometric skills in Elementary Education: from where the teaching-learning process of geometry from the first grades of general education should start and to which it should contribute.

2.3. Actions and operations for the development of geometric skills in Elementary Education

The skills of recognizing geometric objects, tracing and/or constructing, arguing geometric propositions and solving geometric problems, as specific skills of geometry in elementary education, require the schoolchild to perform certain practical and intellectual actions and operations as a product of exercise, which may or may not have been automated, converted into habits and skills already acquired by applying previous knowledge.

For this reason, it was considered necessary to propose the most general actions and operations for the geometric skills developed in these grades. This makes it possible to examine how this process occurs in a planned, oriented and organized manner, from the external to the internal plane and put into practice again.

This proposal of actions and operations was made following the modes and procedures for modeling cognitive activity, as indicated by Talízina (1988). From the above, the first of the modes is assumed: the theoretical-experimental; while from the procedures, according to the functions, those that allow the independent analysis of all the particular phenomena that are the object of study are taken; and according to the content, the logical ones are united to the first of the ways, which at the beginning forms isolated actions, which are later united. In this way, it can be adapted to all spheres of cognitive activity. The proposal is reflected in the following tables.

Table 2. A proposal of actions and operations for the development of the geometric skill of recognizing geometric objects.

Geometric Skills	Actions	Operations
Recognize geometric objects	1. Perceive	Interact with the geometric object, visually and tactilely.
	2. Analyze	Decompose the geometrical object into each of its parts, to know its essential properties.
	3. Relate	Discover the links that exist between that geometric object and others.
	4. Identify	Select the geometric object within a given number of objects, based on its essential and/or general properties.

Table 3. A proposal of actions and operations for the development of the geometric skill of tracing and/or constructing.

Geometric Skills	Actions	Operations
Trace and/or construct.	1. Observe	Visually perceive the geometrical object. Establish correspondence between the geometric object and its properties.
	2. Represent	Materialize the geometric object, from a set of steps, different materials (rods, flat developments, plasticine, etc.), and drawing instruments.
	3. Check	Analyze the validity of the procedure used.

Table 4. A proposal of actions and operations for the development of the geometric skill to argue geometric propositions.

Geometric Skills	Actions	Operations
Argue geometric propositions	1.Recognize	Identify the essential and/or general properties of a geometric object.
	2. Describe	State the essential and/or general properties of a geometric object, using common and geometriclanguage.
	3. Interpret	_Develop conclusions about the elements and relationships of geometric objects.
	4. Explain	_Express reasons that confirm what has been stated, from the essential and/or general properties of geometric objects.

Table 5. A proposal of actions and operations for the development of the geometric skill to solve geometric problems.

Geometric Skills	Actions	Operations
solve geometric problems	1. Interpret	_Understand the problem situation.
	2. Represent	To decompose the geometric object into each of its parts (known and unknown).
	3. Formulate	_Establish a solution procedure (algorithmic or heuristic).
	4. Test	_Demonstrate the validity of the procedure used.
	5. Explain	_Describe the solution procedure used.

It should be noted that this proposal of actions and operations for each of the geometric skills of the first cycle of elementary education should not be followed rigidly, since this process has a systemic character and sometimes the development of a specific geometric skill requires another skill or its actions and operations.

According to what has been analyzed, there may also be cases of students who, in the execution of the activities proposed by the teacher, do not need some of the actions and operations for the development of a specific geometric skill and advance directly to the latter.

2.4. Levels and Indicators to Guide and Evaluate the Geometric Skills Development Process in Elementary Education

The proposed levels to guide and evaluate the process of developing geometric skills in elementary education were determined on the basis of the practical and intellectual actions and operations to be performed by the student and the opinion of the teachers; it was concluded that in order to develop geometric skills, students should: observe concrete objects, experiment with them to discover their properties, and finally, establish relationships between figures, between bodies and between geometric movements.

Based on these ideas, we propose a classification of three levels through which the process of development of skills in each geometric content passes, which, although presented in isolation, for a better understanding, they form a whole and each level is a prerequisite for reaching the higher level. The levels for the development of geometric skills in Elementary Education are:

Level 1 (concretion): the schoolchild, previously guided by the teacher or another group partner, recognizes geometric shapes and movements in the environment, in models and in graphic representations. Observes the differences between figures, between bodies and between geometric movements but cannot properly explain what they are. Reproduces and constructs geometric figures and bodies in an elementary way; makes descriptions using common language, combining geometric symbols and terms; and identifies, in some cases, the procedure to solve geometric problems.

Level 2 (experimentation): the student acts with certain independence. They understand the properties of geometric figures, bodies and movements through manipulation and inquiry activities, which they use to explain the differences. Represents and constructs geometric figures and bodies given orientations in space, applying a procedure. Uses common and symbolic language, based on the appropriation of the term "*geometry*".

Level 3 (abstraction): the student at this level acts with independence. He represents geometric objects mentally and is able to operate with them. He establishes relationships between figures, between bodies and between geometric movements based on their differences and similarities, which he uses in geometric constructions and when arguing the truth value of given propositions. Solves geometric problems that relate: perimeter-area (rectangle) and total area-volume (geometric bodies). Uses the geometric skills acquired in the acquisition of new skills, mathematical content, and/or other subjects; can also provide solutions to everyday situations.

Below is a graphic representation of each of the proposed levels, as well as the relationship between them.

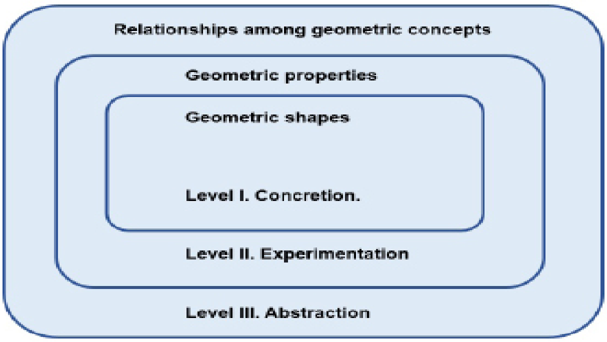

Figure 2. Relationship of the levels for the development of geometric skills in the first cycle of elementary education.

In order to evaluate the level of development of geometric skills, the teacher must also take into account indicators, which are elements that indicate the state in which the student is in the geometric object studied and allow the teacher to move him/her to the desired one based on activities. In this way, the indicators shown in the following tables are suggested; they are also related to the actions and operations that appear in the previous section.

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Table 6. Indicators to guide and evaluate the process of geometric skills development at the concreteness level.

Level I (concretion)			
Geometric skills			
Recognize geometric objects	**Trace and/or construct**	**Argue geometric propositions**	**Solve geometric problems**
1. Determines shapes in the from a given geometric model.	1. Reproduces physical objects using geometric shapes.	1. Matches a given name to the corresponding geometric figure or body.	1. Knows the units of the International System of Measurement: length, area and volume.
2. Determines geometric shapes and mivements from given concrete objects.	2. Reproduces figures (on graph paper, with rods or stencil) and geometric bodies (with rods) from a given model.	2. Names geometric figures and bodies with the corresponding word.	2. Determines the length of the sides of a geometric figure.
3. Determines geometric shapes in the context from a given denomination.	3. Represents points and lines with the necessary plotting tools.	3. Denotes and names geometric figures.	3. Determine the amplitude of angles, perimeter, areas (rectangle and orthohedron) and volume of geometric bodies (orthohedron).
4. Identifies geometric movements (of space and plane) in graphical representations, on graph paper and in different positions.			

5. Identifies figures and bodies in geometric series.			
6. Identifies plane figures by observing geometric bodies from different positions			
7. Relates development with given geometric bodies.			
8. Identifies number of cubes in a stack.			
9. Identifies figures in other compound figures in which only the figure in question appears and are not included within others.			

Table 7. Indicators to guide and evaluate the process of geometric skills development at the experimentation level.

Level II (experimentation)			
Geometric skills			
Recognize Geometrical objects	**Trace and/or construct**	**Argue Geometrical propositions**	**Solve geometrical problems**
1. Selects spatial orientation movements made using the relationships left-right, front-back, up -down, north-south, east-west.	1. Represents straight lines (parallel and perpendicular), segments, and angles.	1. Describes geometric figures and bodies.	1. Compares the volume and capacity of bodies.

2. Selects Geometric shapes and movements in the environment from a given name and vice versa.	2. Traces axes of symmetry in concrete objects and geometric figures.	2. Interprets sentences describing geometric figures and bodies.	2. Master the procedure for making con units of length, area and volume.
3. Analyzes properties of geometric figures, bodies and motions.	3. Represents rectangles and squares on graph paper given the orientations in space.	3. Explains from the domain of geometric properties.	3. Solves geometric problems of perimeter, area and volume, taking into account the properties of geometric figures and bodies.
4. Classifies figures and Geometric bodies, according to their properties.	4. Represents figures (with rods and stencil) and geometric bodies (with rods) taking into account their properties.		
5. Compares geometric figures and bodies	5. Develops formal geometric constructions with tracing instruments (circumference, parallelogram, rectangle and square, etc.).		
6. Identifies how an image is obtained by movement from the original figure.	6. Elaborates geometric figures given a number of elements.		
7. Identifies the number of cubes in a stack represented graphically.	7. Obtains other Geometric figures and bodies from the previous ones.		

8. Identifies necessary quantities of figures and geometric bodies to form other figures and geometric bodies.			
9. Identifies figures in other Composite figures in which only the figure in question appears and they are included one inside the other.			

Table 8. Indicators to guide and evaluate the process of geometric skills development at the abstraction level.

Level III (abstraction)			
Geometric skills			
Geometric skills	**Trace and/or construct**	**Argue Geometrical propositions**	**Solve geometrical problems**
1. Compares the number of times that two types of figures appear without expressing quantities, based on the recognition of the figures in a composite.	1. Represents figures (from others, with rods, stencils and tracinginstruments) and geometric bodies (with rods) where the relationship between concepts is established.	1. Explains geometric situations in the environment that are related to units of magnitude.	1. Solves geometric problems that combine perimeter and area.
2. Compares geometric figures and bodies based on properties.		2. Explains from the mastery of the relationships between geometric concepts.	2. Solves geometric problems where the total area and volume of geometric bodies are related.

3. Identifies in a composition of motions how an image is obtained from the original figure.		3. Expresses reasons why physical objects are related to geometrical properties.	
4. Identifies geometricsituations in the environment units of magnitude.			
5. Relates geometric figures and bodies based on their properties.			

03.

THE METHODOLOGICAL TREATMENT OF GEOMETRIC CONCEPTS IN ELEMENTARY EDUCATION

"Geometric knowledge is not acquired... if at the same time the experience and the mind of the one who receives it are not put into play".

(Canals Tolosa, 1997)

3.1 Methodological Suggestions for the Treatment of the Concepts of Geometric Figures in the First Cycle of Elementary Education

In the teaching of plane geometry, there are three elementary concepts, which are not defined and which are introduced intuitively, always starting from objective reality; these concepts are: point, line and plane.

In the treatment of the concept point, we start from the following notions:

1. The mark made with a pencil.
2. A grain of sand.

3. The head of a pin.
4. The place where a lace is stuck.
5. The corner of the house, where three wall edges meet.

In the introduction of this content, fundamental aspects should be highlighted, such as:

1. The points are represented with a cross and are denoted with capital letters in print.
1. The existence of many points (infinite, although this term is not used).
1. The difference between the geometric entity point, its representation and its name.

The concept of point is closely linked to that of straight line, since it is said that in a straight line there are infinite points. Some notions to take into account when introducing this concept are the following:

1. Power lines.
2. An extended rope.
3. The edge of the ruler.

What is essential in this content is that the students recognize that there are different types of lines, from their representation in various positions (horizontal, vertical or oblique), among which straight lines are.

They should also know that straight lines are denoted with a lowercase letter or two capital letters when two points stand out on the line. It is necessary that schoolchildren master that straight lines cannot be measured; that, like points, they can be drawn as many as desired, and the correct way to check whether a line is straight or not is with the use of the ruler.

In the first grades of elementary school, schoolchildren begin to become familiar with the ruler, which they not only use to draw straight lines, but also to draw other geometric figures, which they will later learn about. It is necessary that teachers, from this moment on, begin to inculcate in schoolchildren habits of cleanliness and precision in the tracing of figures and in the use of drawing instruments.

As part of the relationship that exists between the concept of point and line, one of the main objectives of this content is that schoolchildren recognize that from a point an infinite number of lines can be drawn, and given two different points, one and only one line can be drawn (relation of incidence "... passes through..".).

Schoolchildren must also be able to understand that there are points through which a line does not pass, that these points are not on the line, and that given any three points on a line, there is one and only one of them that is located between the other two (relationship of order "... is located between... and..".).

Another important issue in this content, is that the students perceive that points are determined in the place where two straight lines of plane figures join and where three or more join in geometric bodies.

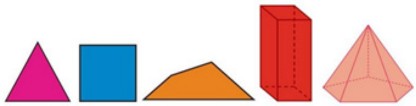
On the other hand, it should be considered that the tracing of straight lines through one and two points, is a prerequisite for the tracing of the concept of ray and segment.

At the end of this content, the geometric notions of parallelism and perpendicularity should be introduced, starting with examples from life, such as the strings of a guitar in the case of parallel lines, and the consecutive edges of the walls of the classroom, in the case of perpendicular lines.

In these contents, we insist that the students understand that all lines that do not cut are parallel, while some lines that cut are perpendicular; the latter can be checked by placing the short sides of the bevel to see if they coincide with the lines that cut. In addition to the above, schoolchildren should master the (algorithmic) procedure for drawing parallel and perpendicular lines using the ruler and the bevel or two bevels, which will be a prerequisite for drawing parallelograms.

For the deepening of this content, activities such as the following can be oriented:

1. Plot two points and denote them.
2. Identify the number of points that stand out in each case:

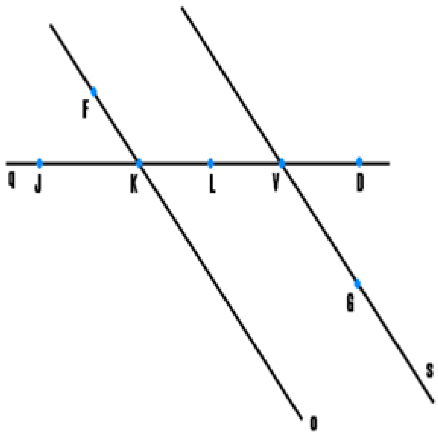


3. Plot a line c and the points K, J and M such that:

a) Points K and J do not belong to the line c.

b) Points K and M belong to the same line.

4. Look at the following figure and select all the correct propositions:



...... Point K is not between points L and V.

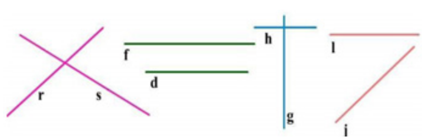
...... Point J is between points Q and K.

...... The line s does not pass through points V and G.

...... Point V is between points L and D.

...... The points F and K are located on the line q.

5. Observe the following pairs of lines and determine in which of them parallel and perpendicular lines have been represented:

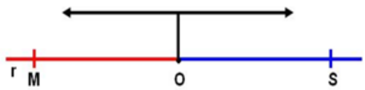


6. Observe the following figure and determine the greatest number of parallel and perpendicular segments:

In the treatment of the ray concept, it should be specified that in a straight line every point determines two ray lines, being that point the origin of the ray lines. It is convenient to fix in the schoolchildren the idea that the ray cannot be measured because they are unlimited from their origin, and for two rays to be opposite, they must meet the condition of having the same origin and be determined on the same line.

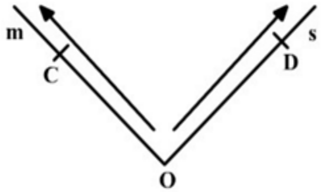
Example 1

In this case, the rays OM and OS are opposite rays because they have the same direction and opposite directions; they are determined on the same line and have the same origin.

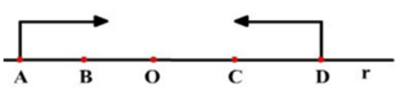


Example 2

In this example, we have represented ray lines (OC and OD) that have the same origin, but they are not opposite ray lines, because they are not determined on the same line.



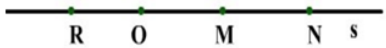
Example 3



In this situation, it should be emphasized that the rays AB and DC are not opposite rays, although they have opposite directions, because they are located on the same line, but they do not have the same origin.

In addition, exercises of recognition of ray lines in compositions of points that are determined on the same line can be proposed, which tend to fix the idea that every point on a line determines two ray lines and that if one continues determining points on the line, one always obtains twice as many ray lines.

Example 4



In this case, we find: 2 . 4 = 8 ray lines.

In the introduction of the concept of segment, it is insisted on:

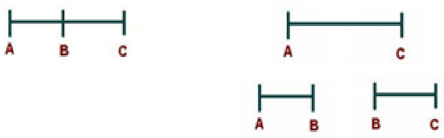
1. A segment is the part of a straight line bounded by two points, including these points.
2. The segment is a figure bounded by its ends.
3. To distinguish what is to denote and to name.
4. A segment is determined by two different points; that is why different letters should be used to denote them, never the same one.
5. Segments are determined on the sides and edges of figures and geometric bodies.
6. Overlapping segments that coincide are equal and have the same length.

It is important to emphasize that the counting of segments in a composition is a prerequisite for the recognition of figures included within others, of the same type and of different figures.

To fix these ideas, exercises such as the one below can be proposed:

1. Indicate the number of segments found in this figure:

To recognize the number of segments found in this figure, you can analyze the figure as a whole and decompose it into each of its parts.



In this way, three segments are obtained: $\overline{AB}$, $\overline{AC}$ y $\overline{BC}$.

Another way in which one can recognize the number of segments that this figure has is by combining the points of the figure, two by two, as described by the principle of intra- and inter-subject relationship of geometry teaching, following the following procedure:

1. The first point is fixed as the first end of the segment, and this point is combined with each of the following ends according to the established order.
2. Then the second point is fixed, and the same operation is repeated.

Three segments are obtained in the same way. $\overline{AB}$, $\overline{AC}$ y $\overline{BC}$.

To fix this content, activities such as the following can be done:

1. Analyze the following propositions and say whether they are true (T) or false (F). Argue for each case.

a) .... Segments and ray lines have origin and end.

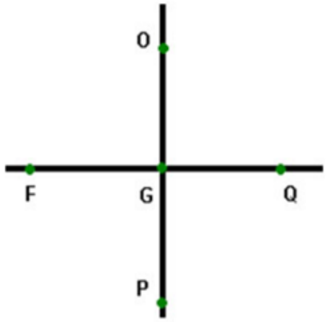
b) .... Each of the parts into which a point divides a line is called a ray.

c) .... The ray, like the straight line, cannot be measured with a ruler.

d) .... Any point on a line determines a ray.

e) .... The point that divides a line in two is called the vertex of the ray.

2. Look at the following figure and select, marking with an X, the correct proposition.

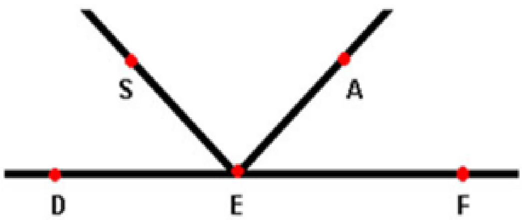


a) .... The ray GF and the ray GQ are not opposite.

b) .... The ray GO and ray GP are opposite.

c) .... The rays GF and GO are opposite.

3. Look at the following figure and select, marking with an X, the correct propositions.

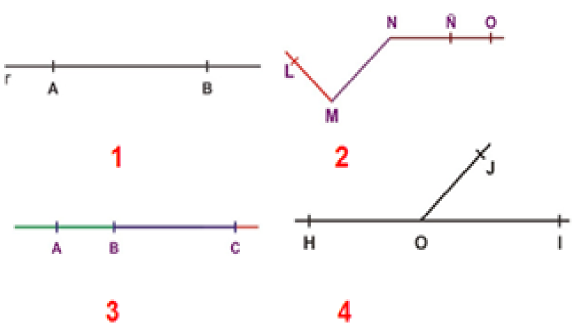


a) .... The rays ES and ray EA are not opposite.

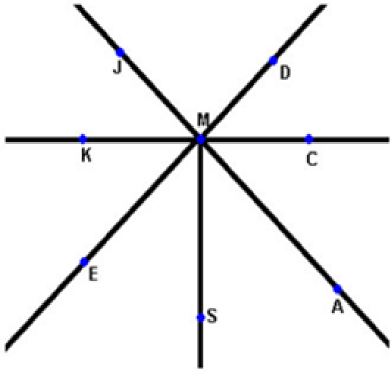
b) .... R ray DE and ray FE are opposite.

c) .... R ray ED and ray EF are opposite.

4. Look at the following figure and tell the number of ray lines that are determined in each case.



5. Given the following figure, select, by marking with an X, the number of ray pairs that are opposite.



a) ....6 pairs of ray-straight lines

b) ....7 pairs of ray-parallel lines

c) ....8 pairs of ray pairs

In the plane concept, all the previous contents must be kept in mind, remembering that the plane, together with the point and the straight line, is one of the basic concepts of geometry since any geometric object can be synthesized from them.

Some notions to introduce this concept are shown below:

1. A soccer field.
2. The floor of a house.
3. The page of a book.
4. The wall of a house.

In the same way, experimental activities can be developed, as proposed by the principle of using concrete objects and other teaching aids, in which the students can recognize in models of geometric bodies flat surfaces and indicate in them the faces that are in intersecting planes (they always have an edge in common) and the faces that are not in intersecting planes, that is, the parallel ones. In the case of round geometric bodies, the flat surfaces of the cone and the cylinder can be identified in their bases, clarifying in the latter that only their bases are in parallel planes. These same activities can also serve as a basis for the appropriation of the unlimited character of the plane.

The concept of the half-plane can be worked on through practical activities: drawing a line on a sheet of paper and folding it from that line. In this way, the plane represented by the sheet of paper is divided by the line into two regions, or half-planes. This line is called the edge of the half plane. It is important for the teacher to emphasize that if two points are in opposite half planes, the segment determined by them cuts the edge; otherwise, it does not cut it; and that the points located on the edge of the half plane belong to both half planes.

To fix the contents of plane and half plane, schoolchildren can perform activities such as:

1. Draw a line s and the points L, M, R, N and D so that:

a) L and M are located on the same half plane.

b) R and N are on the edge of the half-plane.

c) L and D are located in opposite half-planes.

2. Draw a line r and segments from the points B, L, M, N, R and S so that:

a) Segment RS has its ends in the same half-plane.

b) Segment BL cuts the edge of the half plane.

c) Segment MN does not cut the edge of the half plane.

d) Segment RS and segment MN are not in the same half plane.

At the end of this content, the teacher can guide the students to reach the conclusion that in the same way that a point on a line determines two rays, being the point the origin, a line in a plane determines two half planes, and this line is its edge. Also, both the ray and the half-plane are bounded on one side (by the point in the ray and by the edge in the half-plane), but unbounded on the other and extend as much as desired.

At the end of the content of plane and half plane, the following conclusions can be drawn:

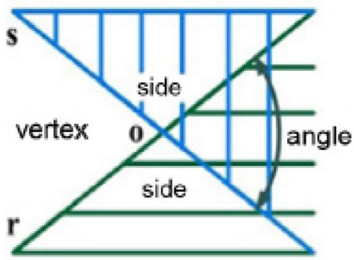
1. The point, the line and the plane are unbounded.

2. Both straight lines and planes intersect or are parallel.

3. The straight lines intersect at a point, while the planes intersect at a straight line.
4. Each point on a line determines two ray lines; the obtained ray lines are opposite.
5. Each straight line in a plane determines two half-planes; the obtained half-planes are opposite.

The introduction of the concept of angle at the end of the cycle, should be motivated by the contents of straight lines, rays, planes, and half planes. The students should be instructed to draw, together with the teacher, two lines that intersect at a point and highlight in different colors the two half-planes that they determine, as shown in the figure below.

Example 5



Next, it will be specified that the part or region of the plane doubly striped, which is common to both half-planes, where the intersection point and the straight lines that are the edges are included, is called an angle.

At this point, the elements of this figure must also be analyzed: (1) vertex, which is the point of intersection of the edges of the two half-planes; (2) sides, formed by the straight lines that limit the doubly striped region; and (3) interior points, those that are common to the two half-planes that are not on their edges.

In order to fix this concept, it is convenient to carry out experimental activities in which the schoolchildren find angles in the known polygons and in the environment. Subsequently, the forms of notation to be used should be introduced: (1) three capital letters, with the letter of the vertex in the middle; (2) a number, which is usually placed in the inner region next to the vertex; and (3) a single capital letter at the vertex.

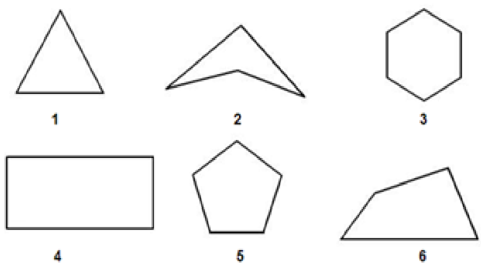
Once the concept has been introduced, it is necessary that the students become familiar with the graduated semicircle and master the procedure for measuring and drawing angles of different amplitudes (from 00 to 180^0). After the presentation and use of the instrument, the schoolchildren should arrive, in addition to the above, at the following conclusions:

1. Right angles measure 90°, and their sides coincide with the short sides of the bevel, since they are perpendicular.
2. The right angles measure 180°, and the ruler can be placed on both of them at the same time, because their sides are opposite half-straights.

In the exercise of this content, both measurement and tracing activities should be proposed; it should also be related to the clock and fractions, which constitute one of the practical utilities of this concept.

Some activities, such as the following, can be used for the development of this content:

1. Observe the following geometric figures and determine for each case the number of angles.



2. Determine, for each case, the approximate measure of the angle formed by the hands.

3. What time does the following clock mark if the hour hand indicates 2 and the minute hand has traveled an angle of 90^o, starting from the hour on the hour?

4. Draw two angles of 40° and 115°. Denote them.
5. At which of the following times will the hands of the clock form a right angle? Mark with an X the correct answer. Note the direction in which the hands of the clock turn, and always start from the hour hand to the minute hand.

......3:30 p.m ......3:15 p.m

......3:00 p.m ......3: 20 p.m

At which of the following times will the hands of the clock form a flat angle? Mark with an X the correct answer. Note the direction in which the clock hands turn, and always start from the hour hand to the minute hand.

......9:00 p.m ......9:10 p.m

......9:15 p.m ......9:30 p.m

The introduction of the polygon concept starts with the concept of a polygonal line and the types of polygonal lines: open and closed, where the portion of the plane limited by a closed polygonal line, including this one, is called a polygon. It is important for schoolchildren to know, from the first grades, the elements of polygons: vertices, sides (consecutive and opposite), and angles (the latter in the fourth grade); although the term polygon is not worked from first to third grade. Also, its classification according to the number of sides.

The triangle is one of the polygons that is approached in the first cycle of Elementary Education. In the first grades, it is emphasized that those triangles that coincide when superimposed are equal and that this is the polygon with the least number of sides. Schoolchildren can form the idea that although the objects are not triangle-shaped, they can be found on the faces of many of them.

Among the representatives of the polygon concept, there is also the quadrilateral, to which a large part of this content is devoted. In an intuitive way and indicating formative characteristics by presenting to the students examples of the same class and other classes, the following concepts can be elaborated: quadrilateral, trapezoid, parallelogram, rectangle, rhombus and square.

Comparing triangles and quadrilaterals will allow schoolchildren to recognize the common characteristics and differences between these figures. They recognize that both triangles and quadrilaterals are polygons. However, they differ in their number of elements.

When dealing with the different quadrilaterals, the relationships between the different concepts should be established explicitly. In this way, at the end of all the content in the analysis of the classification of quadrilaterals, the schoolchildren should reach the following generalizations:

1. All polygons that have four sides are quadrilaterals.

2. Quadrilaterals may or may not have parallel, opposite sides.

3. Quadrilaterals that have at least one pair of parallel opposite sides are trapezoids.

4. Quadrilaterals that have their opposite sides parallel are called parallelograms.

5. Parallelograms are also trapezoids.

6. Parallelograms that have their consecutive sides perpendicular are called rectangles.

7. Parallelograms that have equal sides are called rhombus.

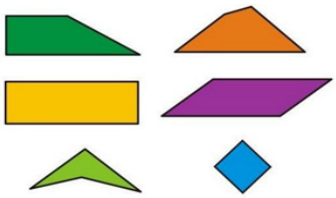
8. A rectangle that has equal sides is called a square.

9. The rhombus that has its consecutive perpendicular sides is called a square.

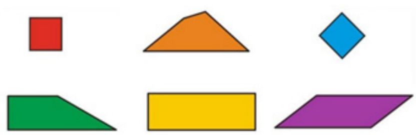
These generalizations are expressed in the principle of abstract geometrical thinking, since it can be observed how in the treatment of these concepts, special importance is given at all times to the relationship between each one of them.

This can be obtained as a conclusion by the students themselves if they carry out activities such as the following:

1. Analyze the following geometric figures and determine if they are all quadrilaterals. Argue.



2. Observe the following quadrilaterals and select the ones that satisfy the following property: “It has its opposite sides parallel”.



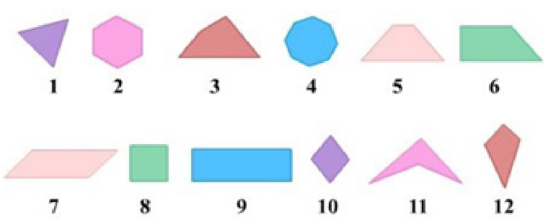
3. Compare the number of triangles and rectangles and point out the correct answer:

...... There are more triangles than rectangles.

...... There are more rectangles than triangles.

...... There are as many triangles as rectangles.

4. Given the following geometric figures, recognize:



a) Quadrilaterals..

b) Rectangles...

c) Trapezoids...

d) Rhombuses...

e) Parallelograms..

5. Analyze the following propositions and state whether they are true or false. In each case, give an argument.

a) Some squares are rectangles.

b) Every rhombus is a square.

When introducing the concepts of circumference and circle, the elements of the circumference (center, radius and diameter) should be pointed out, in addition to highlighting the relationship between the equality of the radii and the equality of the respective circumferences. In the same way, as when schoolchildren began to draw figures with the template and the rest of the drawing instruments, it is very important that schoolchildren become familiar with the compass, its parts and use.

Although the concepts of circumference and circle are closely linked because the elements of one correspond to the other. The student must reach the conclusion that the curved geometric figure formed by all the points of the plane that are located at equal distance from an interior point, called the center, is called the circumference; while the geometric figure formed by all the interior points of a circle, including this one, is called circle.

In the development of the contents of geometric figures, it should not be forgotten that geometric constructions have important functions within the geometry class, since the mastery of this skill allows the student to elaborate geometric figures and bodies as representatives of any concept, to understand their properties, and to expand the geometric vocabulary, when describing the procedure used.

In their geometric learning, schoolchildren must reproduce geometric figures from given models using different materials (paper, cardboard, plasticine, wire, wood, lace, scissors, colors or tempera) and tracing instruments.

These activities, besides being related to the principle of the contextual character of geometric contents, the use of concrete objects and other teaching means, the intra- and inter-material relation of geometry teaching and abstract geometric thinking, can be linked to the principle of continuous support in historical knowledge; it would only be enough for the teacher to find the right moment and make known historical curiosities to motivate learning in schoolchildren.

They can also carry out activities of geometric constructions through the use of teaching aids such as the tangram and the geoplane, which provide a procedure for their construction, using geometric notions of equality and parallelism.

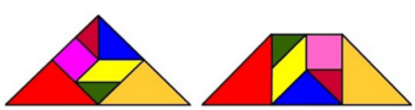
In this way, students will be able to represent polygons with different means and procedures on the basis of data given orally, in writing or graphically, starting from the mastery of geometric properties.

Some exercises that can be used to check the achievement of the objectives, together with those found in the textbooks and workbooks, are the following:

1. Trace on graph paper the following geometric figure:

2. Represent on the geoplane the following quadrilaterals:

3. Construct with the Tangram the following geometric figures:



4. Construct with the tracing instruments a parallelogram ABCD.
5. Draw two circles of radius 2 and 3 cm, respectively. Denote their center.

6. Cut out the following figures. Then, with a single cut, form in each case:

a) A quadrilateral that is not a parallelogram.

b) A parallelogram that is not a rectangle.

3.2. Methodological suggestions for the treatment of plane movements in the first cycle of Elementary Education

The treatment of geometric movements in the first cycle begins with the observation and perception of elements of the surrounding world, since it is from reality, as expressed by the principle of the contextual character of geometric contents, that the process of abstraction and generalization of this knowledge takes place.

In the third grade, the introduction of geometric movements—translation, reflection and rotation—must be done, taking as a reference the actions of translating, reflecting and bending, performed in the environment. To deal with the aforementioned terminology, it is suggested only to mention that objects in space and figures in the plane are translated, reflected or rotated.

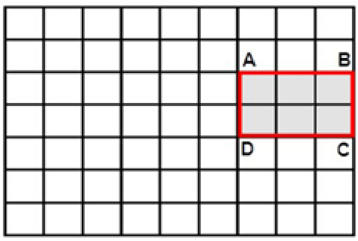
To introduce the concept of translation, it is suggested to present examples from everyday life, such as a car driving along a straight road; the opening and closing of a sliding door; and a child crossing the street in a straight line to get to school.

On the other hand, spatial orientation activities can be carried out. These spatial activities will help schoolchildren understand that they can move in any direction. They must also be able to perceive that objects do not change their shape or size after moving.

In order for them to understand that the same thing happens when moving geometric figures on a flat surface, the teacher can help them with the use of the grid paper or the geoplane, guiding them to move or observe the number of grids or units, respectively, in which the figures have been moved, as shown in the following examples:

Example 1

On the graph paper, move the rectangle ABCD from its current position six grids to the left:



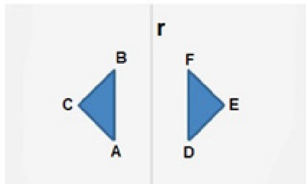
If an example similar to this one is used for the introduction of this geometric movement, it should be internalized by the schoolchildren that each point of that figure moves at the same distance. The schoolchildren should also know that the place where the rectangle was initially located is called the original figure, while the place it occupied after moving is called the image figure. By means of this movement, each point of the original figure has been made to correspond to a single point in the image figure.

Similarly to how we proceeded with the concept of translation, the concept of reflection must be introduced. Some examples of this movement in the environment are: a child looking in a mirror; the image of the moon on the surface of calm water; and the projection of a movie in a theater.

From the demonstration of these and other examples, the students should come to the conclusion that in all cases the properties of shape and size remain the same. However, the reflection of the image is inverted with respect to the physical position of the object.

Subsequently, a situation of the plane must be analyzed. The most traditional way to introduce this movement is from the use of a sheet of paper, folded in half, where a figure is drawn on one of its parts with tempera, and before it dries, both parts are joined together, pressing strongly, as illustrated in example 2, although the triangles are not left with their sides exactly straight:

Example 2



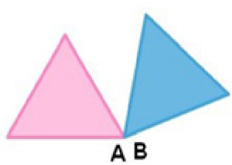
It is important for the students to note that, as in the previous movement, each point of triangle ABC (initial or origin figure) corresponds to a single point in triangle DEF (final or image figure).

Analogously to how the two previous concepts were introduced, we must proceed with the introduction of the rotational motion. Some examples of this movement in the medium should be indicated, such as the following: the star of an amusement park; a spinning top while dancing; and the hands of a clock.

In all these cases, it must be shown that all the points of these objects rotate at the same distance from their axis and that they can only rotate in two directions: right or left.

To exemplify this movement in the plane, a geometric figure must be presented that is rotated by hand about one of its points. In this case the triangle ABC has been rotated, starting from point B (which remains fixed), to the right, to which corresponds, as an image, after the movement, the triangle DEF.

Example 3



After performing this movement, the students must realize that all the points of this figure rotate at the same distance. Another important idea to understand is that in this movement, the shape and size of the figure remain the same. However, its orientation varies.

Once the study of geometric movements has been completed, the students should reach the following conclusions:

1. When a figure is moved in the plane, a figure equal to the original one is obtained.
2. By means of a movement, each point of the original figure corresponds to a single point of the image figure.
3. If two figures are equal, it is because one is obtained from the other by means of a movement.

To deepen these contents, exercises such as the following can be proposed:

1. Analyze the following situations and determine what happens in each case:

a) A train through a line where there are no curves.

..... Turns. ..... It moves. ..... It reflects.

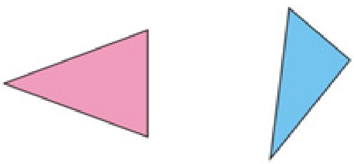
A child in front of a store window.

..... Turns. ..... Moves. ..... It reflects.

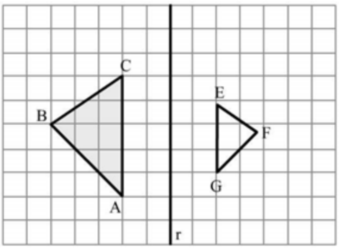
b) The wheels of a car around its axles.

..... Turns. ..... Moves. ..... It reflects.

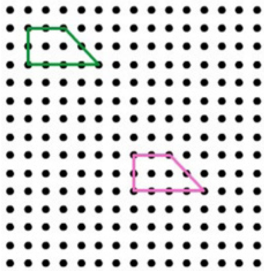
2. Look at the following triangles and say whether the following proposition is true or false: "The blue triangle is not an image of the other triangle because of a motion".



3. Analyze whether triangle ABC is an image of triangle EFG by motion. Argue.



4. Select the procedure you can follow to translate in the geoplane the pink figure, from where it is, to the position occupied by the green one. Mark with an X the correct answer.

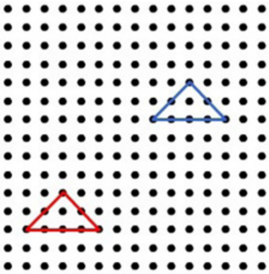


..... I move 7 units up and 6 units to the right.

..... I move 6 units to the left and 7 units up.

..... I move 7 units down and 6 units to the right.

5. Select the procedure not to follow to move the red triangle on the geoplane from where it is to the position occupied by the blue triangle. Mark with an X the correct answer.



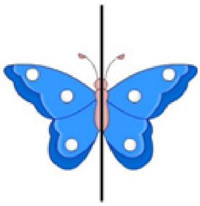
..... I move 7 units to the right and 6 units up.

..... I move 2 units up, 7 right, and 4 up.

..... I move 6 units up and 6 to the right.

The treatment of the contents of symmetrical figures should be done intuitively in the process. It is suggested that the symmetrical figures be introduced beforehand and later on the pairs of symmetrical figures with respect to an axis. Initially, the students should be presented with situations in which they must trace or fold figures to determine if they are divided into exactly two equal parts, as shown in the following example.

Example 4



The teacher can also exemplify what happens when a mirror is used. In each case, it should be clear that symmetrical figures are

In each case, it should be clear that symmetrical figures are obtained and that this imaginary line is called the axis of symmetry.

In the treatment of this content, the potential of the environment must be used. Schoolchildren should be able to recognize objects that have symmetries in them, such as many leaves on trees, some animals, the human body, certain buildings and other objects. These activities will allow them to conclude that a figure is symmetrical if an imaginary line can be found that divides it into two equal parts or if, by placing a mirror on half of the figure, the reflection and the half form the complete figure.

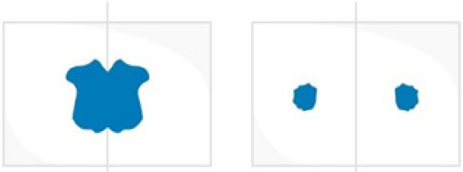
Next, students should be given the opportunity to determine which of the known geometric figures (triangle, rectangle, square, circle and other polygons) are symmetrical and the axes of symmetry they have, through experimental activities or by mastering the properties of these figures.

These activities will allow the students to draw the following conclusions:

1. All polygons do not have axes of symmetry.
2. The triangle that does not have equal sides does not have axes of symmetry.
3. The triangle that has two equal sides has an axis of symmetry.
4. A trapezoid that has equal non-parallel sides has an axis of symmetry.
5. There are three parallelograms with axes of symmetry: the square, the rectangle and the rhombus.
6. Polygons with equal sides have the same number of axes of symmetry as sides.
7. The circle has infinite axes of symmetry, which correspond to its diameters.

In order to continue deepening in this content, the students should be presented with a situation in which they observe what happens when a sheet of paper is folded in half, after dropping a drop of paint on it, in the center and in a place near one of its ends, as shown in the following example.

Example 5



In both cases, a pair of equal figures is obtained, because, when they are superimposed, from the symmetry axis, they coincide. It should be noted, in both examples, that each point of the original figure corresponds to a point in the image figure, and they are located on a straight line perpendicular to the axis of symmetry and at equal distance from it; however, in the first case it is observed that there are points that are on the axis and coincide with it.

To exercise this content, it is suggested to carry out activities similar to the following ones:

1. Mention the names of objects in the environment that have axes of symmetries.
2. Observe the following alphabet and classify each letter as:

a) Symmetrical.

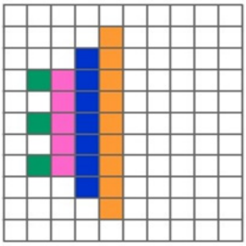
b) Not symmetrical.

c) Only one axis of symmetry.

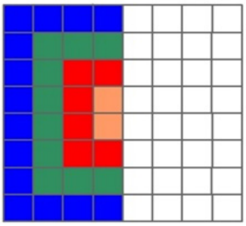
d) More than one axis of symmetry.

3. Complete each of the following figures so that symmetric figures are obtained.

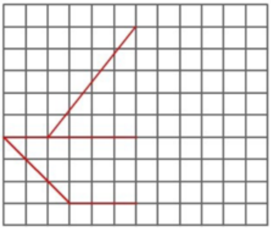
a)



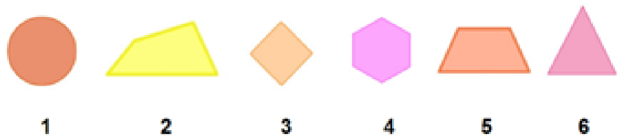
b)



c)



4. Given the following geometric figures:



a) Determine which ones are symmetric and the number of axes of symmetry they possess.

5. Construct two quadrilaterals having two and four axes of symmetry.

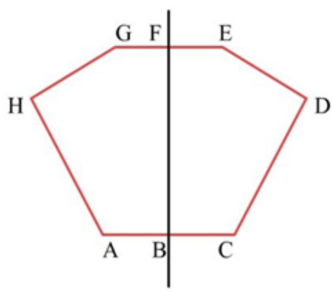
6. Analyze the following propositions and state whether they are true (T) or false (F). Argue for each case.

a) .... All rectangles have 4 axes of symmetry.

b) .... Some trapezoids have more than one axis of symmetry.

c) .... There are geometric figures with infinite axes of symmetry.

7. Observe the following figure and establish the relationship of equality between the sides and angles that correspond with respect to the axis of symmetry FB.



3.3. Methodological Suggestions for the Treatment of the Concepts of Geometric Bodies in the First Cycle of Elementary Education

In order to continue expanding the geometric knowledge of schoolchildren, geometric bodies limited by flat surfaces and round bodies are introduced by means of concrete objects. Among the geometric bodies limited by flat surfaces, the orthohedron, the cube, the prism and the pyramid are worked on.

Generally, concepts are formed through two ways: inductive and deductive. The following is a series of actions to introduce the geometric concepts of orthohedron and cube from the inductive way in the execution stage:

1. Observation of geometric properties in the environment: place different objects on a table (ball, shoebox, Parcheesi dice, book, pen and spool of thread) and instruct the students to place on the table the objects that meet the following property: they are limited only by rectangles.
2. Making empirical generalizations: the students, under the guidance of the teachers, reach the conclusion that the objects that are limited only by rectangles have the shape of an orthohedron. We insist that, from this stage on, the students learn that objects limited only by squares also share these characteristics.
3. Realization of theoretical generalizations: this action is related to the previous one, because once the students identify the common essential characteristics (invariable) in the objects, the teacher elaborates the orthohedron concept with the help of the students: The orthohedron is a geometric body limited by six faces that are rectangles, the cube being part of the extension of the concept. Previously, the students should know that the faces in geometric bodies are the geometric figures that limit them.
4. Determination of other properties of the concept: through experimental activities carried out by the students with the help of geometric models, other properties of the orthohedron and the cube are known, such as the number of vertices, edges and relation of position of their opposite faces.
5. Search for new objects with this geometric shape in the environment: once the students know the essential characteristics of the orthohedron, they will be able to look for objects in the environment that have the shape of an orthohedron and explain the reason why they have this shape.
6. Mastery, systematization and self-assessment of geometric skills: the teacher can indicate a series of

activities that serve to enhance and evaluate the passage of their students through each of the levels of development of geometric skills, where the student, according to his characteristics, is able to relate the contents of the orthohedron and cube with others that he already has and apply it in the solution of everyday situations.

The mastery and systematization of geometric skills can be achieved using different teaching media, ranging from concrete objects to educational software. In this action, feedback from the students is indispensable, since each one of them should be offered the possibility of self-evaluating the level of development of the acquired geometric skills.

It is also suggested to introduce the concept of a prism and to conclude the study with the generalization by the students that the orthohedron and the cube are prisms because they have a pair of parallel and equal opposite faces, called bases, and their other faces are rectangular. In all prisms, it is true that their number of lateral faces coincides with the number of sides of the polygon that forms their bases.

In order to fix these ideas, it is considered necessary to organize activities to assemble and disassemble flat developments of these geometric bodies with rods and plasticine, which, in addition to being observed by the students from different positions, take into account the relationship between these concepts.

On the other hand, when approaching the pyramid concept and its elements, the following conclusion can be reached: "The number of faces of the pyramid is equal to the number of sides of the polygon that constitutes its base".

At the end of the study of geometric bodies limited by flat surfaces, students should be able to establish similarities and differences between prisms and pyramids:

Similarities:

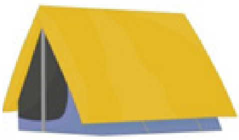
1. They are geometric bodies bounded by plane surfaces.
2. The number of sides of the polygon that limit their bases or base (in the case of the pyramid) determines their number of faces.

Differences:

1. The prism has rectangular faces, and the pyramid has triangle faces.
2. The prism has two bases, and the pyramid has only one.

Some activities that can be used in the treatment of these contents are as follows:

1. The tent house that is represented has the same shape as:



a) ...... A cube

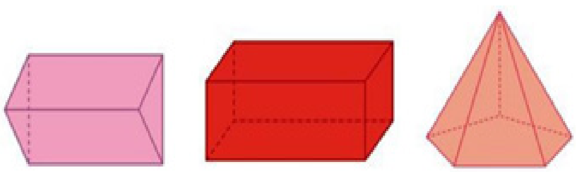
b) ...... An orthohedron

c) ...... A prism with a triangular base

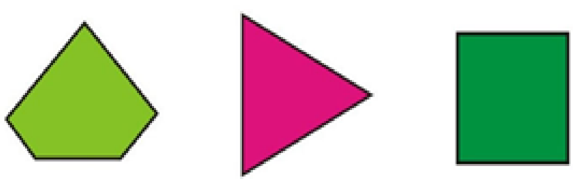
d) ...... A pyramid with a square base.

2. Link the following geometric bodies with the geometric figure observed if viewed from the right:

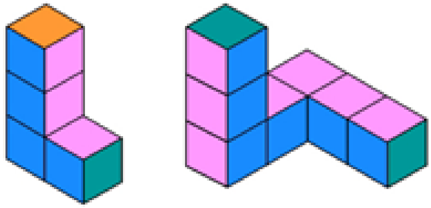
Bodies



Plane figures



3. How many cubes must be added to the body on the left to obtain the body on the right? Mark with an X the correct answer.

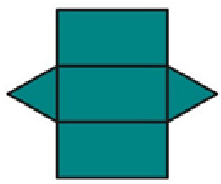


a) ......3 cubes.

b) ......7 cubes.

c) ......4 cubes.

4. Select which geometric body is formed by assembling the following plane development:



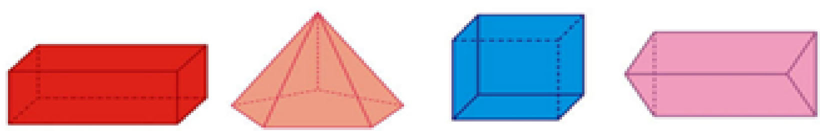
a) ...... A prism with a triangular base.

b) ...... A cube.

c) ...... A prism with a rectangular base.

d) ...... A pyramid.

5. Construct the following geometric bodies with rods:



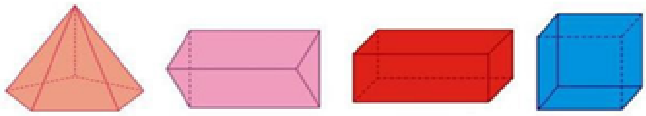
6. Construct with rods a prism other than a cube.

7. Analyze the following propositions and state whether they are true or false. Argue.

a) Some orthohedra are prisms.

b) All pyramids have a triangular base.

8. Observe the following geometric bodies and say how they are similar and different:



9. Analyze the following questions and give an answer to each one of them, discussing them with your classmates.

a) Why do the roofs of many of the houses have the shape of triangular prisms?

b) Why do boxes have the shape of an orthohedron?

Among the round geometric bodies, the cylinder, the cone and the sphere are discussed. It is convenient to use as a variant for the introduction of these geometric concepts the recognition of objects in the environment with these geometric shapes, in addition to identifying their elements and properties, through practical activities.

When the cylinder is taught, it is important to emphasize that it is limited by two flat surfaces that are circles, considered its bases, and by a curved surface.

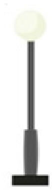
When presented with cone-shaped objects of the medium, the students, in models, should know its difference with the cylinder: it has only one flat surface that is also a circle, considered its base.

In the treatment of the sphere, it should be noted that, unlike prisms, pyramids and the rest of the round bodies, it has no bases or faces and is completely formed by a curved surface. Schoolchildren must have reached this conclusion when they analyzed the surfaces of the bodies when the flat content was taught.

So that schoolchildren do not tend to confuse the concepts of circle and sphere, it is important that they understand the difference between the two concepts: The circle is contained in a plane and the sphere is not. The analysis of the similarities and differences of these bodies will enable schoolchildren to know the reason why they are round bodies. Activities of assembling and disassembling plane developments of these geometric bodies (cylinder and cone), which can be observed from different positions, can be used to teach these contents.

Suggestions of types of exercises that can be used to check the achievement of the objectives:

1. The top of the streetlight that is represented has the same shape as:

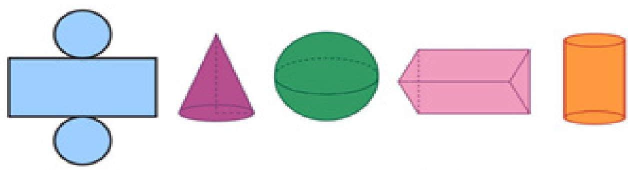


a) ...... A cone

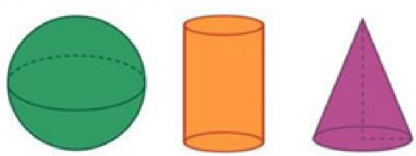
b) ...... A sphere

c) ...... A cylinder

2. Circle the geometric body that can be formed with these elements:



3. Observe the following geometric bodies and tell how they are similar and different:



4. Analyze the following questions and give an answer to each of them, discussing them with your classmates.

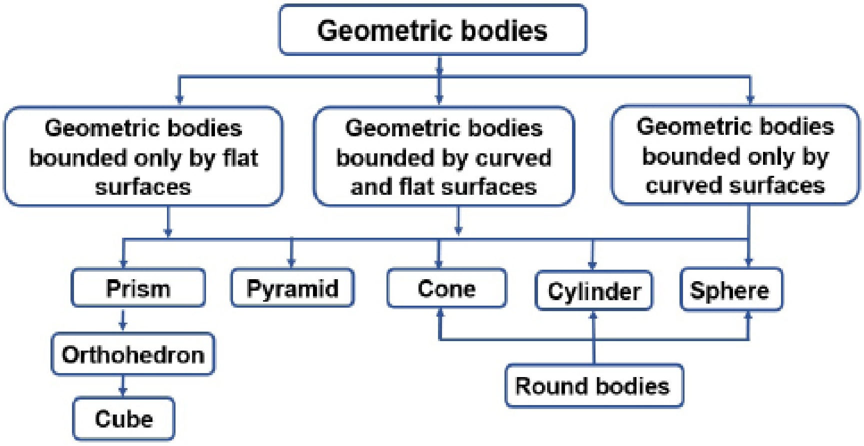
a) Why are the pipes that transport water and other liquids shaped like cylinders and not like orthohedra?

b) Why is the funnel shaped like a cone?

c) Why do many rolls of yarn have a cylindrical shape?

d) Why do balls roll easily?

The following is a graphical representation of all the geometric bodies discussed in the cycle, as well as the relationship between them:



3.4 Methodological Suggestions for the Treatment of Planimetry in the Second Cycle of Elementary Education

In the treatment of geometric contents in the second cycle of Elementary Education, a very important aspect is the consolidation of the fundamental concepts of Geometry such as point, straight line, ray, segment and some relationships in which these concepts intervene. It is recommended to exercise the identification of these elements in the medium and in geometric figures, insisting on the argumentation from the essential characteristics of each one, with the objective of continuing the development of geometric thinking and to create the preconditions for the introduction of the concept of angle.

For this reminder of plane and half plane, a sheet of paper (or the plane of the teacher's table, or the blackboard) can be used. Here it should be emphasized that these representations only give us an idea, since the planes are not limited (in these concrete examples it can be said that they are not limited by the edges of the paper, the table or the blackboard), but that one should imagine that they extend as long as one wishes.

If you use the example of paper, the idea of a half-flat naturally arises from the folding of the sheet along a folding line. This form is useful because it also allows us to remember the notion of a straight line. In this case, it should be emphasized that the line is not limited either but can be extended as much as you want, and that each time a line is drawn in the plane, two half-planes are formed.

This can be illustrated on the blackboard, highlighting how straight lines are denoted (by two capital letters or by a single lowercase letter), as well as the relationships that can be established between them (highlighting, in particular, parallelism and perpendicularity).

In the treatment of the concept angle, the following skills should be worked on:

1. Recognize its elements: vertex, sides.
2. To denote angles.
3. To name angles.
4. Identify angles in other figures.
5. Measure angles in sexagesimal degrees.
6. Estimate the measure of angles in sexagesimal degrees.
7. Draw angles with a given measure.
8. Classify angles according to measure in sexagesimal degrees.

The first methodological step in the treatment of the concept is the elaboration of the concept. For its elaboration, two ways can be followed: the inductive or the deductive.

In the inductive way, 2 variants are suggested:

1. Schoolchildren are instructed to represent 2 half-planes whose edges intersect at a point.

- Subsequently, they are asked to shade the part common to both, including the edges.
- The students are instructed to represent 2 other half-planes whose edges intersect at a point.

- They are asked to shade the union of the 2, including the edges.
- Finally, they are informed that both the (common) intersection and the union of the half planes are called angles, and the definition is specified.

2. A teaching aid can be used as described below:

- A piece of poster board or cardboard on which a line has been drawn and one of the two half-planes has been lined in color.
- A piece of plastic or transparent paper on which the same has been done, but using another color.
- Place one on top of the other, making the lines coincide, and place a pin on a point of the line that is near the center of the paper.

The plastic or transparent paper is rotated, and thus it is clearly seen how the angle emerges as the intersection of two half-planes (doubly striped region) and the angle as a union.

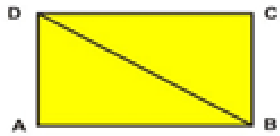
Finally, the definition of angle as the intersection or union of 2 half-planes whose edges cut or intersect and its elements are highlighted:

1. Vertex (point of intersection of the edges of the two half-planes).
2. Sides (straight lines that limit the doubly striped region).

In a deductive way, the definition is presented, and then it is exemplified, and its elements are reported.

The second methodological step is the development of skills in the identification of angles in the environment and in the different geometric figures and bodies studied, recognizing their different elements and ways of naming and naming.

Here it is important to insist that the most used form is that of three letters and that the letter of the vertex is placed in the center. It is also necessary to emphasize that sometimes it is not convenient to use the notation of a single letter because, for example, in the following figure, A is the vertex of 3 angles.



The angles that are determined are: $\sphericalangle ABC$, $\sphericalangle ABD$ y $\sphericalangle DBC$, note that all three have the same vertex.

Once the concept of angle and its different forms of notation have been introduced, we are ready to measure the amplitude of an angle. This can be done by analogy with the segment:

Once the concept of angle and its various forms of notation have been introduced, one is in a position to move on to measuring the amplitude of an angle. This can be done by analogy with the segment:

1. Presentation of the instrument (ideally, each schoolchild should have one in his/her hands) and explanation, with the active participation of the schoolchildren of its description, highlighting:
2. In the segments, it is useful to measure their length or extension (their length), and for this purpose the graduated ruler is used, in centimeters and millimeters.
3. With the ruler, it is also possible to trace segments with a given length.
4. Can the opening or amplitude of an angle also be measured? With what instrument? With what unit of measurement?

The solution to this problem is the graduated semicircle. The methodological steps recommended for this work are the following:

1. Presentation of the graduated semicircle.

The parts into which it is divided. Introduce the denomination of degree for each of the 180 parts into which the semicircle is divided. Insist on the notation used and inform about the subdivision into minutes and seconds, although the latter is not essential.

The center, clarifying that it must be sought on the line where 0^0 and 180^0 are and specifying that it is the midpoint or center.

2. Explanation of the placement of the semicircle to measure angles.

In this step, 2 important aspects must be taken into account:

- How to use the semicircles that do not have double graduation and those that do.
- How to measure an intersection angle and how to measure a union angle.

To measure any angle, the vertex coincides with the center and one of the sides with the ray with origin in that center and passing through 0^0 or with the one passing through 180^0. The other side of the angle must be between 0^0 and 180^0.

If it is an intersection angle and one side of the angle is made to coincide with 00 then the measure of the angle is that indicated by the semicircle, but if one side of the angle is made to coincide with 180^0 then the measure of the angle is 180^0 minus that indicated by the semicircle.

If it is an intersection angle and one side of the angle is made to coincide with 0^0, then the measure of the angle is that indicated by the semicircle; but if one side of the angle is made to coincide with 180^0, then the measure of the angle is 180^0 minus that indicated by the semicircle.

If it is a union angle and one side of the angle is made to coincide with 0^0, then the measure of the angle is 360^0 minus that indicated by the semicircle; but if it is an intersection angle and one side of the angle is made to coincide with 180^0 then the measure of the angle is 180^0 plus that indicated by the semicircle.

If the semicircle has a double graduation, it should be explained to him that one always starts with the reading of 0^0, either the one on the left or the one on the right, never with 180^0, and the corresponding measure is read following the direction in which the other side is found. If it is an intersecting angle then the measure of the angle is the one indicated by the semicircle, but if it is an intersecting angle then the measure of the angle is 180^0 plus the measure indicated by the semicircle.

For the development of skills in measuring angles, it is suggested that you measure any angles, but also pairs of angles, angles of triangles and quadrilaterals, in order to prepare preconditions for the respective properties that will be studied later.

The other skill to work on is that of tracing angles with a given measure. The development of this skill is a precondition for the development of the measuring skill. It will only be necessary to emphasize the drawing of angles with a measure greater than 180°. For this, it is necessary to explain that the measure is decomposed into 180° plus another measure that is always smaller than 180°. For example: 300° = 180° + 120°. First we draw a plane angle (180°) and then we draw an angle that measures 120°.

To practice both the measurement and the drawing, the angles should be proposed in an increasing order of difficulty:

1. Angles whose measures are less than 180° (intersection angles).

   Angles whose measures are multiples of 10 (30°, 40°, and 130°).

   Angles whose measures are multiples of 5 and not 10 (35°, 45°, and 115°).

   Angles having any amplitude (58°, 103°, and 178°).

   Angles that are in different positions.

   Angles that are inserted in given figures.
2. Angles whose measures are less than 180° (intersection angles).
3. Angles whose measures are greater than 180° and less than 360° (union angles).

   Angles that are in different positions.

Another important skill to keep in mind when dealing with different magnitudes is estimation. This skill can be worked on in both measuring and tracing exercises.

The above skills serve as a basis for the introduction of the classification of angles according to their measure. This is an important aspect for one of the criteria of triangle classification. In the classification of angles, the angles measuring 0°, 90°, 180°, and 360° serve as a limit for classification.

For the achievement of the systematization of the skills to be developed, integrative exercises can be proposed, such as:

1. Draw an acute angle and:

a) Point out its elements.

b) Denote it with 3 capital letters.

c) Name it.

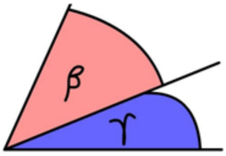
d) Estimate its measure.

e) Determine its measure with the semicircle.

An aspect of deepening of the concept angle is the relative one to relate the position that several angles keep among themselves. In these relationships, the following concepts to be studied are highlighted:

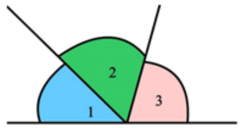
- Consecutive angles.
- Angles opposite at the vertex.
- Pairs of angles formed between 2 straight lines cut by a third one.

When approaching the concept of consecutive angles, it is important to emphasize that 2 angles are consecutive if they only have in common the vertex and one side.

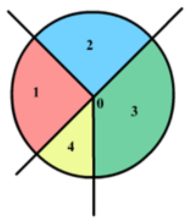


It is convenient to give as counterexamples pairs of angles that not only have in common the vertex and one side or that only have in common the vertex or that apparently have in common one side.

The concept of consecutive angles must be extended to more than 2 angles with the idea that "one following the other", i.e. they are consecutive 2 by 2. Among the examples of more than 2 consecutive angles, we must emphasize the consecutive angles on one side of a straight line and the consecutive angles around a point.



Consecutive angles on one side of a straight line



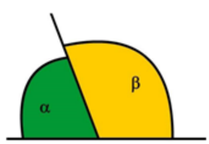
Consecutive angles around a point

Teachers should get students to recognize that: $\sphericalangle 1 + \sphericalangle 2 + \sphericalangle 3 + \sphericalangle 4 = 360^0$ and conclude that in general it is true:

"Consecutive angles around a point form a complete angle and add up to 360".

In the case of consecutive angles on one side of a straight line, they should recognize that: $\sphericalangle 1 + \sphericalangle 2 + \sphericalangle 3 = 180^0$ and from several examples conclude that in general it is true that:

"Consecutive angles on one side of a straight line form a plane angle and add up to 180^0".



Among the examples of consecutive angles on one side of a straight line should appear some with only two angles, taking advantage of the opportunity to introduce the concept of adjacent angles:

"Two consecutive angles on one side of a straight line are called adjacent angles".

As an extension for the teachers, it is considered to deepen the structure of the definitions. There are different types of definitions. The definition of adjacent angles is an existential definition, specifically called by some authors real or objective; this type of existential definition has the following structure:

Concept to be defined = Superior concept (generic) + Invariant characteristics.

The concept to be defined is known by definiendum and that which defines it by definiens. In the above definition, adjacent angles are the definiendum (concept to be defined), two consecutive angles are the generic concept, and on one side of a straight line, the type characteristic. It is important to note that the same concept can be defined by different expressions, i.e., what is defined, known by definiendum, can be associated with a set of different terms and relations, named definiens.

For example, the concept of adjacent angles can also be defined as follows:

1. Two consecutive angles whose union is a straight angle are called adjacent angles.
2. Two angles that have only one side in common and whose union is a straight angle are called adjacent angles.

In expression 1, the superior or generic concept "consecutive angles" is maintained, and the expression of the characteristic is replaced by an equivalent one. In the case of 2, the generic concept is replaced by its respective definiens and the expression of the invariant characteristic is maintained. The 3 definitions are equivalent since the same representatives correspond to them. It is important to have these theoretical aspects of definitions to guide more accurately their understanding, as well as to contribute to the rigor of mathematical language by schoolchildren.

After elaborating on the concept of adjacent angles, schoolchildren should be led to recognize that:

“The amplitudes of adjacent angles add up to 180^0”.

The above proposition is a mathematical truth that can be proved from the previous concepts; therefore, it constitutes a theorem. It is convenient to express the theorems in the form of implication, that is, in the form “If p then q”, where p is the premise or sufficient condition for q and q is the thesis or necessary condition for p. In this case it is: “If two angles are adjacent, then their amplitudes add up to 180^0”. Therefore, that 2 angles are adjacent is the premise or sufficient condition for their amplitudes to add up to 180^0 and that their amplitudes add up to 180^0 is a necessary condition for 2 angles to be adjacent.

The argumentation (demonstration) of the property of adjacent angles must be made in conjunction with the schoolchildren in such a way that the essentials are indicated:

1. If they are adjacent they are consecutive to one side of a straight line (by definition).
2. If they are consecutive on one side of a straight line, they add up to 180^0 (by property of consecutive angles on one side of a straight line).

In the previous analysis, it is necessary to emphasize:

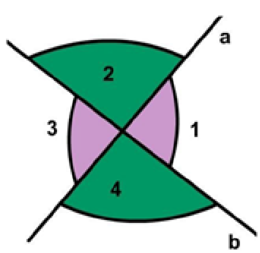
1. The property of adjacent angles seems very obvious, but for its argumentation one starts from the premises, and with the use of a definition and a previous property one arrives at the thesis.
2. In mathematics, many statements can be found, such as the property of adjacent angles, whose veracity must be established from definitions and other properties already known, and this process of argumentation is called proving.
3. Such true statements or propositions are called theorems, and that in them something is given as information, the premise or hypothesis, and what is sought, which is the thesis.

Teachers should make schoolchildren understand that not always when the amplitudes of 2 angles add up to 180^0, they are adjacent. Based on the previous analysis, the students should be asked to determine the truth value of the following proposition (reciprocal of the adjacent angles theorem): “If the amplitudes of 2 adjacent angles add up to 180^0, they are adjacent”. With a counterexample, it can be stated to be false. It should be noted that the theorem of a theorem is formed by interchanging the premise with the thesis.

It is also good for teachers to know how to form the counterproblem of a theorem (the premise is the negation of the thesis of the original theorem and its thesis the negation of the premise of the original), as well as that the counterproblem of a theorem is always true. In the case of the adjacent angles theorem, without the need to mention the term counterfactual to the schoolchildren, they can be asked to analyze the truth of the following proposition: “If 2 angles do not add 180^0 then they are not adjacent”.

Although it is not intended that schoolchildren make a study of logical aspects, it is convenient that teachers move their logical thinking, and this analysis of reciprocal and counter-reciprocal contributes to this end. But in order to promote this end, teachers must know elements of logic.

After dealing with consecutive angles, another pair of angles is treated, but they only have the vertex in common. Teachers can present several pairs of angles with only the vertex in common, among which there must be at least 2 pairs whose sides are respectively opposite half-straights, that is, they form 2 straight lines.



The concept of opposite angles at the vertex can be defined as follows: “The angles that are formed when two straight lines intersect are called opposite at the vertex”. This definition of opposite angles by the vertex is genetic because it expresses how opposite angles are obtained by the vertex.

It should be noted that when drawing 2 lines that intersect at a point, 2 pairs of opposite angles are formed at the vertex:

In addition, you can take advantage of the representation of angles opposite by the vertex to recognize that adjacent angles are also formed by asking: do you identify other pairs of angles? How many pairs of adjacent angles are formed?

To obtain the vertex opposite angles theorem, one can proceed in several ways; one of them consists of measuring the angles and observing the results. A second way is to give the measure of one of 4 angles to find the amplitudes of the others, applying the property of the adjacent angles. For example:

If $\measuredangle 1 = 78^0$, calculate the amplitudes of angles 2, 3 and 4.

$\measuredangle 1 + \measuredangle 2 = 180^0$ because they are adjacent angles.

$78^0 + \measuredangle 2 = 180^0$ by substituting.

$\measuredangle 2 = 180^0 – 78^0$.

$\measuredangle 2 = 102^0$.

Analogously, we obtain that $\measuredangle 3 = 78^0$ and $\measuredangle 4 = 102^0$.

By any of the 2 ways, the students will be able to conclude that: “The amplitudes of the opposite angles by the vertex are equal, or simply the opposite angles by the vertex are equal”.

It is convenient to analyze the following aspects with the schoolchildren:

1. Implicative form of the theorem, highlighting the premise and the thesis.
2. Argumentation (proof) of the theorem.
3. Formation of the reciprocal and counter-reciprocal and determination of their veracity, respectively of both.

The second way of the theorem search allows the understanding of the argumentation.

Since 1 and 2 are adjacent because they are consecutive to one side of the line.

$\sphericalangle 1 + \sphericalangle 2 = 180^0$ by the property (theorem) of the adjacent angles.

$\sphericalangle 3 + \sphericalangle 2 = 180^0$ by the property (theorem) of adjacent angles.

$\sphericalangle 1 + \sphericalangle 2 = \sphericalangle 3 + \sphericalangle 2$ because both sums are equal (180^0).

Then $\sphericalangle 1 = \sphericalangle 3$ because both sums add up to $\sphericalangle 2$.

It is important to note that there is another way to obtain the theorem by motions, and that also guarantees its truth. For this, an auxiliary material can be used (cardboard and transparent paper superimposed), making a central symmetry with the center at the vertex of the angle. In this movement the vertex is a fixed point and the straight lines (sides) of the angles are transformed into opposite straight lines, respectively, therefore the angles are equal.

At the end of the introduction of the pairs of adjacent and opposite angles by the vertex, it is convenient that the students establish similarities and differences between these pairs, emphasizing that in both cases the angles have a common vertex and that at least one side of an angle is the ray of one side of the other angle in each pair. As for the amplitudes, the opposites by the vertex are always equal, and the adjacent ones always add up to 180^0, and they are only equal when 2 right angles are involved.

At the beginning of the treatment of the pairs of angles formed between 2 straight lines cut by a third one, it is important to emphasize that now we will study pairs of angles that do not have a common vertex, and it is necessary as a precondition that the students know the different regions that are formed when tracing the 3 straight lines:

1. The region between the two straight lines is called internal, and the one outside of them is called external.

2. The third line that cuts the other 2 is called secant and divides the plane into two half planes, one on one side of it and the other on the other side.

An activity that is very useful to train schoolchildren in the characteristics of the angles according to the regions mentioned above is to denote the 8 angles that are formed given any 2 straight lines cut by a secant and ask questions such as:

1. Which angles are in the outer region, and which are in the inner region?
2. Which angles are in the same half plane with respect to the secant?
3. Which pairs of angles are in different regions, and which are in the same region?
4. Which pairs of angles are in the same half-plane, and which are in different half-planes with respect to the secant?

Next, the schoolchildren are asked to select the pairs of angles that are in the following combinations of positions:

1. In different regions and on the same side of the secant.
2. In the same regions and on different sides of the secant.
3. In the same regions and on the same side of the secant.

In this way, the concepts of corresponding angles, alternate angles and conjugate angles are introduced. A very important aspect in the elaboration of these concepts is that the non-straights cut by the secant must not always be parallel, since this condition is not essential for these concepts.

For the fixation of these concepts, it is recommended to do exercises of identification of the studied pairs, including the adjacent angles and the opposite angles by the vertex.

Once these concepts are assimilated, we move on to the fundamental point of the treatment of these pairs, which is the study of the theorems of each one when the short lines are parallel. This is an essential premise for the fulfillment of the properties expressed as theorems.

Several ways can be used to search for the theorems:

1. Measurement with a graduated semicircle.
2. The movements with the concrete material. (For the corresponding ones by means of a translation and the alternate ones with a central symmetry.) The property of the conjugate angles is obtained starting from the properties of the previous couples, never from a movement, because they are not equal.

Also, after knowing the property, for example, the corresponding angles, it is possible, by means of an exercise, knowing the amplitude of one of the angles, to find the amplitude of the remaining angles applying the property of the corresponding angles, the adjacent ones and the opposite ones by the vertex. Finally, the students will observe that the alternate angles have the same amplitude and that the conjugate angles add up to 180^0. Very simple formulations should be selected for the 3 theorems to facilitate the formation of the reciprocals and counter-reciprocals without difficulty. For example:

Theorem:

"Corresponding angles formed between parallel lines are equal".

Reciprocal:

"Equal corresponding angles are formed between parallel lines".

Counter-reciprocal:

"Corresponding angles that are not equal are not formed between parallel lines".

This formation of reciprocals and counter-reciprocals becomes more understandable from writing the theorem in implicative form and highlighting the premise and thesis. In these theorems, it must be emphasized that there are two conditions. The first is that it deals with a certain pair of angles and that they are formed between parallel lines, and for the formation of the reciprocal, the condition of the pair in question is maintained.

The argumentation (proof) of the theorems can be guaranteed from the moment they are obtained, depending on the way used. The treatment of the argumentation must be done with great skill so that the students understand that they are not accepting the veracity of them by a simple statement, but that they are being based on other previously acquired knowledge.

Another essential aspect in the treatment of these pairs of angles is the generalization with respect to any 2 angles formed between 2 parallel lines cut by a secant:

1. If they are classified in the same way (both straight 0 acute or obtuse), then they are equal.
2. If one is classified as acute and the other as obtuse, then they add up to 180^0.

In order to establish the theorems, exercises to calculate angle amplitudes that link all the pairs of angles in different situations should be presented.

The polygon concept has been worked on from previous grades, so it can be seen as a review. The fundamental thing that teachers must achieve is that the students understand the polygon concept and activate their knowledge about the known triangles and quadrilaterals so that they can use them throughout the course of geometry. Particularly, in the case of the triangle, an in-depth study of its elements and properties will be made.

The motivation of the polygon concept, and at the same time its elaboration, can be done in an active way, indicating the students to draw two (three, four, five) consecutive segments in such a way that each one has only one common end (and they are not on the same straight line). From what they draw, the activity can be continued by specifying that they have drawn polygonal lines that can be open or closed.

If no student draws a closed one, some impulse can be given by indicating that in the previous drawing, or in a new drawing, the polygonal line drawn must have in common one end of the first segment drawn with one end of the last one. To conclude this work, the definition of polygon and its elements is specified:

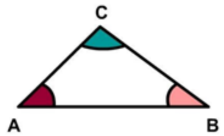
"*The portion of the plane bounded by a closed polygonal line, including it, is called a polygon*".

Elements: sides, vertices, angles and diagonals.

Taking into account the previous definition of polygon, the representation of each polygon must also color its interior, since its interior is also part of it. Regardless of the above clarification, sometimes, in order to minimize the complications of a figure, only its outline (closed polygon) is represented. In any of the variants that teachers use, it is very important that the students recognize the polygons already known to them (triangles and quadrilaterals), which should be used to discuss the classes of these polygons already known and their most characteristic properties.

Among the known representatives is the triangle. This can be used to recognize the triangle as a three-sided polygon and highlight its elements. It is important to insist on the sides, to name them, as well as the interior angles and to propose to the students triangle-tracing exercises so that they themselves denote them and name their vertices, sides and angles.

Triangles have three sides, three vertices and three angles.

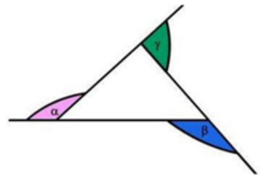


In this triangle, the vertices are the points A, B and C; the sides AB, BC and AC; and the angles ABC, ACB and BAC.

It is necessary to introduce the relationship "opposes" between sides and angles, as well as the concept of exterior angle:

A opposes side BC, B opposes side AC, and C opposes side AB, respectively.

The exterior angles of a triangle have as vertex a vertex of the triangle, as sides: one side of the triangle and the prolongation of the other at the considered vertex.



It is important to note that at each vertex there are 2 exterior angles that are equal because they are opposite angles at the vertex, but when we talk about the exterior angles of a triangle, we only refer to 3, one at each vertex.

Having reaffirmed the concept of a triangle, its classification must be elaborated according to the measure of its sides and angles.

For the above-mentioned, there may also be different methodological variants:

1. The schoolchildren are offered the classification so that they can experimentally test the statements made by measuring sides and angles.
2. They are given a set of triangles in which the three types to be studied are presented (this can be done by means of a worksheet). They can be instructed to measure the sides (the angles) and draw conclusions about the relationships between their sides (their angles). Teachers can give impulses to guide attention to what is wanted.

Then they can regroup them according to the same characteristics:

1. There are triangles that have equal sides and others that do not have equal sides.
2. There are triangles that only have acute angles; there are others that have 2 acute angles and one right angle, and others that have 2 acute angles and one obtuse angle. All have at least 2 acute angles.

After forming the classes of triangles according to the characteristics of their sides (of their angles), the teachers will give them the name that each class receives. In the case of the classification according to their sides, it should be clear that equilateral triangles are a special class of isosceles triangles, i.e., every equilateral triangle is isosceles, but not every isosceles triangle is equilateral.

The worksheet should be conceived so that schoolchildren, starting from impulses given by teachers, can reach:

1. Establish relationships between both classifications:

- No equilateral triangle can be right-angled or obtuse-angled.
- Isosceles triangles can be acutangles, rectangles or obtuse angles.
- Scalene triangles can be acutangles, right angles or obtuse angles.

2. Establish relationships between sides and angles.

- Scalene: The three angles are different.
- Isosceles: The two angles opposite the two equal sides are also equal.
- Equilateral: The three angles measure the same (they are equal).

Therefore, the students will be able to conclude from this practical work a new mathematical truth: "Different (equal) sides oppose different (equal) angles and reciprocally".

To motivate triangular inequality, schoolchildren can be given rods of different sizes to test whether triangles can be formed with them. With some trio of rods, it will not be possible to form a triangle, so it seems that there are conditions for the lengths of the sides, which must be found.

To find those conditions, teachers can use a trio with which a triangle cannot be formed and compare the sum of the lengths of 2 sides with that of the third side and ask what must happen. For them, you can continue comparing the sums in other trios of rods; you can also return to the worksheet that was used for the classification of triangles. In this way, you can conclude: "In every triangle, the length of each side is less than the sum of the lengths of the other two sides".

To motivate the theorem of the interior angles of a triangle, the worksheet used for the classification of triangles can be used again, in this case by adding the amplitudes of the 3 interior angles of each triangle. The students will observe that if they round to the nearest multiple of 10, they get that they add up to 180^0, i.e., “The sum of the amplitudes of the interior angles of a triangle is 180^0”. It should be noted that it is possible that not in all cases 180^0 will be obtained due to possible inaccuracies when measuring with the semicircle. There are other variants for the search of this assumption using the concrete material, but in all of them it is necessary to awaken in the schoolchildren the need to argue it (demonstrate it). Although it can also be obtained by means of the following exercise that guarantees its veracity:

If in a triangle ABC it is satisfied that CD // BA and E a point on the ray BC. Prove that $\sphericalangle BAC + \sphericalangle ABC + \sphericalangle ACB = 180^0$.

If the requirement of the previous exercise is replaced by Prove that $\sphericalangle ACE = \sphericalangle ABC + \sphericalangle BAC$,

Another statement about the angles of a triangle can be obtained, in this case the exterior angle theorem. Measuring and working with the concrete mathematics can also be used as a way so that schoolchildren compare the amplitude of each exterior angle with those of the interior angles until they arrive at: “In every triangle, the amplitude of each exterior angle is equal to the sum of the amplitudes of the interior angles not adjacent to it”.

With the same concrete material, it can be asked to measure the 3 exterior angles, which add up the measurements, and the students will be able to observe that approximately each sum is equal to 360^0, because the following assumption of the exterior angles of a triangle can be obtained: “The sum of the amplitudes of the exterior angles of a triangle is equal to 360^0”. In this way, the study of the triangle is concluded, the exercise should be directed to the fixation of the theorems by means of exercises of argumentation and calculation of amplitudes of angles. It should be used to systematize the properties of the pairs of angles.

The quadrilaterals and their properties constitute another important aspect in the treatment of polygons. Although schoolchildren have been learning in previous grades. The fundamental thing that teachers must achieve is that they master the characteristics that define each one and that they know how to establish relationships between concepts, incorporating in them a more precise vocabulary in terms of mathematical terms. For example, it is not acceptable for schoolchildren to say that Quadrilaterals are figures that have 4 sides because the figure is not the appropriate generic concept for the concept of quadrilateral; there are figures that have 4 sides and are not quadrilaterals.

We must insist on the fundamental criteria used to classify quadrilaterals:

1. Those that do not have parallel sides (trapezoids).
2. Those with parallel sides (trapeziums).

Considering this classification criterion, the second group is the one that should be given special attention. However, within the trapezoids, the one that has 2 pairs of consecutive equal sides but not 3 equal sides (symmetrical trapezoid) stands out. The trapezoid that is not symmetric is called an asymmetric trapezoid and is the most general quadrilateral.

In the second one, a partition of 2 subsets can be made, taking into account how many pairs of parallel sides they have, but all are called trapezoids because they have at least one pair of parallel opposite sides.

1. The first subset are those with only one pair of parallel opposite sides (most general trapezoid).
2. The second subset are those that have two pairs of parallel opposite sides, called parallelograms.

The most general trapezoids can be classified according to the length of their non-parallel sides in an isosceles trapezoid (non-parallel sides of equal length) and according to whether one of the non-parallel sides forms right angles with the bases (rectangular trapezoid).

The trapezoids of the second group, specifically called parallelograms, can be subdivided into 3 subsets: one of them has its 4 right interior angles called rectangles, another one has its 4 equal sides called rhombuses, and the third subset does not fulfill any of the previous properties (more general parallelogram). But the first 2 subsets have a non-empty intersection, i.e., there are parallelograms that have their 4 right angles and 4 equal sides; these parallelograms are both rectangles and rhombuses and, in particular, are called squares.

Teachers must make the students internalize the relationships that are implicit in the previous explanation. So that they can classify the quadrilaterals according to the given criterion and recognize that:

1. All quadrilaterals are 4-sided polygons.
2. There are quadrilaterals that are not trapezoids.
3. There are trapezoids that are not parallelograms, but all parallelograms are trapezoids.
4. There are parallelograms that are not rectangles, but all rectangles are parallelograms.
5. There are rectangles that are not squares, but all squares are rectangles.
6. There are parallelograms that are not rhombuses; all rhombuses are parallelograms.
7. There are rhombuses that are not squares, but all quadrics are rhombuses.

Another important aspect to keep in mind when dealing with polygons is that schoolchildren should identify which ones are symmetrical with respect to a line and/or which ones are symmetrical with respect to a point. Determining the number of symmetry axes they have.

3.4.1. The treatment of the perimeter of polygons in the second cycle of Elementary Education

The fundamental thing that teachers must achieve is that schoolchildren understand the concepts of perimeter and area of polygons, as well as develop skills with their calculation in formal exercises, exercises with text and problems related to practical life.

The concept of polygon, skills in measuring and tracing, and units of length are preconditions for schoolchildren to understand the new knowledge.

The concept of perimeter can be introduced in a very simple way; for example, schoolchildren are instructed to measure the outline of a book, the teacher's table, or any other flat object in the classroom and to add up the lengths of all sides. This is enough to define the concept of perimeter as the sum of the lengths of the sides of a polygon. It can also be done from a life situation:

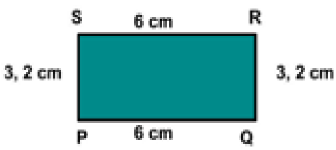
What is the minimum number of meters of thread to tie an orthohedron-shaped box? (A real-life situation can be presented in the classroom).

The idea is not that the students learn particular formulas; the idea is that they understand the concept of perimeter and apply it to all polygons, using the properties of particular polygons (isosceles triangles, equilateral triangles, parallelograms, and rhombuses) to facilitate their calculation. It is also important that the data be given in different units of length so that they have to make conversions.

For example:

Calculate the perimeter of a rectangle whose sides measure respectively 6, 0 cm, and 3.2 cm.

Note that in the previous exercise, the figure is not represented; the students have to identify the corresponding figure and its properties. It is convenient that they represent the figure and place the data, taking into account the properties of the rectangle.



P = a + b + c + d

P = 6,0 cm + 3,2 cm + 6, 0cm + 3,2 cm

P = 18,4 cm

It is also possible to consider:

P = 2 . 6,0 cm +2 . 3, 2 cm P = 2(6,0cm + 3,2 cm)

P = 12,0cm + 6,4cm P = 2 . 9,2 cm

P = 18,4 cm P = 18,4 cm

These last 2 forms can be expressed by means of the following respective formulas, but it is necessary that the schoolchildren master them, although the most important thing is that they understand them:

$$P = 2 \cdot a + 2 \cdot b \qquad \text{and} \qquad P = 2\,(a + b)$$

The requirements for the solution of the problems have already been pointed out above; however, it is not contradictory that in this case teachers teach the students to organize the data and elements necessary for the solution of each problem; this will help to determine the type of figure or difficulty of the problem itself (i.e., data expressed in different units, unnecessary data, insufficient data, etc.).

3.4.2. The treatment of the area of the rectangle and the orthohedron in the second cycle of Elementary Education

For the treatment of the calculation of area, the following preconditions must be guaranteed: the concept of surface and its units. To introduce the concept of surface area, teachers should, starting with representatives, emphasize that geometric figures have a certain “extension” and that this extension is represented by the part of the plane limited by its edge or contour. They can also use the classroom floor, the walls, and the blackboard, and let the students understand that they already know how to calculate the perimeter of some of the figures, that is, the length of their edge or contour, but that they do not know how to obtain the “extension” or “surface” that is contained within their perimeter.

This can serve as motivation for the work that follows, since it should be concluded that they will learn new units that will allow them to express the amount of surface area of a plane figure, as well as to calculate the surface area of some known geometric figures.

Different figures can be presented on graph paper, including a rectangle, so that the students can count the number of squares inside and conclude that this is a way of “measuring” the surface area of these figures: by covering them with little squares, which in this case are the squares of the paper.

They should know that the number of unit squares covering the figure is called the area of that figure.

Subsequently, the students should work with several rectangles whose surfaces have 12 squares of 1 cm^2. For example, one of 1 cm, 12 cm, another of 3 cm, 4 cm, and finally one of 2 cm, 6 cm.

The students are instructed to measure with their ruler one side of one of the inner squares of the rectangle and conclude that it measures 1 cm. It will be pointed out that the square with a side of 1 cm has an area that will be called a square centimeter, which will be written 1 cm^2.

You must then ask yourself: How many square centimeters are there in each rectangle? Then the area of each rectangle is 12 cm^2. It is written $A = 12\ cm^2$. It should be noted that the rectangles are not equal, but have the same area.

Next, to obtain the formula for calculating the area of a rectangle, ask the teachers: How to obtain the area of a rectangle without counting the 1 cm^2 squares, if it is known that its width is 2 cm and its length is 4 cm.

The students will be able to conclude that the area of a rectangle can be calculated by multiplying the lengths of the width and the length. They are informed that the formula is:

A = a. b, where a is the width and b is the length.

Subsequently, the concept of square meter (m^2) should be elaborated, for which 2 ways are suggested: teachers can take a square meter previously represented on cardboard or indicate the students to draw on a cardboard (it can be collaborative work) a square that has one meter of side. In any of the ways, it is necessary to point out that the square contains 100 cm^2, and it is said that the area of this square is 1 m^2. It must be concluded that 1 m^2 = 100 cm^2.

Teachers should emphasize that the square meter is the fundamental unit of surface units and insist that schoolchildren have a clear idea of the area it occupies. This work can be concluded by measuring the length and width of the classroom floor (or blackboard) and calculating the area in square meters.

The students should realize that each unit of length can correspond to a unit of area.

Based on the above, explain to the students the multiples and submultiples of the square meter, writing them on the chalkboard with their symbol and relationship to the m^2.

Multiples	square kilometer - km^2 = 1 000 000 m^2 square hectometer - hm^2 = 10 000 m^2 square decimeter - dam^2 = 100 m^2
Submultiples	square decimeter - dm^2 = 0,01 m^2 square centimeter - cm^2 = 0,000 1 m^2 square milimeter - mm^2 = 0,000 001 m^2

It should also be explained how nowadays the square hectometer and the square decimeter have adopted another symbol:

- square hectometer (hm^2) = hectare (ha)
- square decimeter (dam^2) = area (a)

It is necessary for schoolchildren to understand how to obtain from each known unit the next higher or lower one and to sum it up in the property:

Each unit is 100 times larger than the next lower and 100 times smaller than the next higher.

From the above, it is inferred that when you want to go from a larger unit to the next lower one, you have to multiply, in this case, by 100 and from a smaller one to the next higher one, you have to divide, in this case, by 100.

Here they can also make use of a means in the form of a scale that, as with the units of length, will make it easier for them to make the conversions, so it is convenient for the students to memorize it.

In order for them to memorize the multiples and sub-multiples of the square meter, you can perform oral exercises of conversion from a larger unit to a smaller one and vice versa. Then you can give any unit and ask for the one that follows it, or the one before it, or all the ones that follow it, or that are before it.

They should also be informed about another unit of area that does not belong to the International System of Units (SI) but is used in our country and indicate that they should read in their textbook its equivalence to the square meter. It should be noted that the cavalry, the hectare and the area are used in the measurement of land and are therefore called agricultural units:

Cavalry (1 cab = 134 202 m²).

Teachers should organize the exercise in such a way that it contributes to the understanding of the concept and calculation of area and the relationship between them. On the other hand, the calculation of total areas of orthohedra should be seen as an application of area calculation. In order for the students to understand what they have to do, teachers can have a cardboard orthohedron ready so that they can develop it. They can prepare several templates with the orthohedron developed, and the students should trace it in their notebooks, measure the edges, and compare the faces by superimposing them, and so on. They can then ask how much paper or cardboard has been used to make the orthohedron, and how is the area of the rectangle calculated?

The first question is a motive to orient towards the objective and make them recognize the importance of this activity in practice. Here the fundamental thing is that they understand that there are three different pairs of faces that are rectangles. For the search for pairs of lengths of each of these rectangles, the schoolchildren can observe that if the dimensions of the rectangles are equal to the lengths of each of the faces, they will be able to see that they are rectangles.

a b c   a . b, a . c   *and*   b . c

Finally, the total area of the orthohedron is calculated as follows.

$$A = 2(a \cdot b) + 2(a \cdot c) + 2(b \cdot c)$$

3.5 Methodological Suggestions for the Treatment of Plane Movements in the Second Cycle of Elementary Education

Teachers should take into account the skills that have been worked on in the first cycle on the movements of the plane; in the second cycle, it is a matter of deepening this content. The study of axial symmetry as a property of some geometric figures and the introduction of the concept of movement, in a not yet rigorous way, allow schoolchildren to acquire a constructive geometric procedure that makes it possible to obtain equal figures in a more precise way.

The fundamental thing that teachers should achieve in schoolchildren is that they recognize figures that possess the property of axial symmetry, as well as their respective axes, obtained by folding paper, tracing and working on graph paper.

As a significant issue, it can be pointed out that the detailed study of axial symmetry as a property of some geometric figures makes it possible to introduce the notion of correspondence between points, which will later be necessary for the treatment of movements as well as the equality of the parts that are corresponding.

There are many ways to introduce the concept of axial symmetry. Teachers can prepare worksheets with triangles and quadrilaterals for students to investigate whether there is a line of folds that divides them into two equal parts. In other cases, they can trace symmetrical figures with the same objective.

They can also use the experience of the drop of ink, i.e., take sheets of paper and drop a small drop of ink or paint on them and then ask them to fold the sheet in half.

This variant is very rich because it gives the possibility that both a symmetrical figure and a pair of symmetrical figures can emerge depending on where the drop of ink is and the folding line that is chosen.

From this first experimental work, we must conclude:

1. A figure is symmetrical if we can find an imaginary line that divides it into two equal parts.

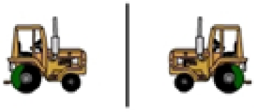
2. The line of folding is called the axis of symmetry.
3. The figure has axial symmetry if there is an imaginary line that divides it into two equal parts.

It is important to note that there are figures that have more than one axis of symmetry.

Among the examples of the practice carried out previously, we must conclude with respect to pairs of figures:

Pairs of figures for which there is an axis of symmetry are called symmetrical figures with respect to an axis.

2. Pairs of symmetrical figures with respect to an axis are equal.



From here, it is recommended to organize an exercise that includes:

1. Recognizing symmetrical figures.
2. Systematize symmetrical triangles and quadrilaterals.
3. Recognize and determine axes of symmetry.

Among the exercises, the following actions can be proposed: complete by tracing the missing part, sketch it following the information given by the grids, recognize symmetrical figures and draw symmetry axes on graph paper as a resource to take advantage of the characteristics of the grids.

Subsequently, the properties of symmetrical points in an axial symmetry are approached, thus giving notions of correspondence between points as a precondition for working with movements.

The fundamental thing that teachers must achieve is that students understand that a correspondence can be established between the points of symmetrical figures, and learn what properties these corresponding or symmetrical points have.

By folding the sheet along the fold line and superimposing the equal parts, it is evident that each point (A, B, C) of one part corresponds to, or overlaps in this case, with a point (A', B', C') of the other part. A pin can be used, and the sheet can be pierced to obtain corresponding points. The important thing is to analyze whether these corresponding points fulfill some property. They can let the students come up with their hypotheses, and whenever necessary, the teachers can stimulate the analysis with some questions:

1. How far are A and A' from the axis? Is it the same for the other symmetrical points?
2. What is the relationship between the segment AA' and the axis, and is it the same for BB' and DD'?
3. What happens with the corresponding of C? Where is C located?

On this basis it must be concluded: "Two symmetric points are perpendicular to the axis of symmetry and at equal distance from it. If they are on the axis, then they coincide".

It is important to report here that it is often customary to denote the symmetrical point of another using the same letter but with a "prime" ('), but that this is not necessarily the case.

After recognizing the property of symmetrical points, the constructive procedure for drawing symmetrical points is discussed. Schoolchildren should learn how to draw symmetrical points with a ruler, bevel and compass and can begin to develop skills in the construction of symmetrical figures, an activity they will continue to perform in the exercise related to reflection in a plane. Teachers should get the students to propose the steps of the procedure, for which they can use prompts through questions.

The image of A is sought:

1. What relationship must exist between the r-axis and the straight line that contains A'?
2. Where must this line pass through?
3. How far should A and A' be from the axis?

Another methodological variant could be to give them the steps, and at the same time that they carry them out, they can analyze why these steps are taken.

The drawing of the perpendicular to r passing through each point can be done through the procedure learned in the previous grades, using a ruler and square (bevel).

As a consequence of the treatment of symmetrical figures can be introduced:

1. The concept of the bisector of a segment and its properties.
2. The concept of the bisector of an angle and its properties.

The concepts of bisector and bisector can be introduced at the same time, starting from the problem:

Are segments and angles symmetrical about an axis?

With templates or paper models conveniently folded, schoolchildren can quickly investigate this and propose their hypotheses about what the axes of symmetry might be in each case.

After this work is completed, explain to the students that these axes have special names and write them on the blackboard. It is important that, in this previous intuitive work, some properties of these two geometric objects are concluded, which are an immediate consequence of what is already known about axial symmetry:

1. The perpendicular bisector is perpendicular to the segment and passes through its midpoint.
2. All the points of the perpendicular bisector (and only them) are equidistant from the ends of the segment.
3. The bisector starts from the vertex of the angle and divides it into two equal angles.

Once these preconditions are met, we can proceed to discuss the procedure for drawing the bisector of a segment with ruler and compass. It is important to note that this same procedure is used to determine the midpoint of a segment. It is important that the length of the radius of the circles to be drawn must be equal to and greater than half the length of the segment.

The drawing of the bisector should be seen as an application of the drawing of the bisector; therefore, the algorithm for its construction can be simplified by reducing it to the drawing of the bisector of a segment.

It is important to relate the concepts of bisector and bisector with the axes of symmetry in the figures studied. For example, if the isosceles triangle has two equal sides, then the vertex that opposes the base is at the same distance from the ends of it, it belongs to the bisector, therefore the bisector coincides with the axis of symmetry and as this axis divides that angle into two equal ones, it must also be bisector. This work with symmetrical figures should not be concluded without first informing the students that there are also many bodies that are symmetrical. In this case they should know that there is no straight line but a plane of symmetry.

The treatment of symmetrical figures is a precondition for approaching the concept of movement and equality. Working with the movements of the plane, without losing its mathematical rigor, cannot neglect the relationship established with life. It is important for teachers to make schoolchildren understand the concept of motion, seen as a special correspondence of points in the plane. They must also understand the properties of the motions, since they will be used later on.

The preconditions on point correspondence were addressed, without elaborating the general concept of correspondence, which is not done, when studying axial symmetry. This concept was limited to corresponding points between figures and must now be extended to any points in the plane.

To begin with, it is convenient to recall some (physical-mechanical) motions that can be made on a plane. To do this, teachers can use models of plane geometric figures and "move" them on the blackboard plane. In this work, it should be specified that:

1. In this movement, each point of the original figure corresponds to a single point in the image.
2. The original figure and its image are the same, since by this movement they do not suffer any deformation either in their shape or in their size.

For the former, they can open small holes in the mode so that they can mark original points and image points on the blackboard. They should take advantage of this to introduce these denominations.

Extending this correspondence between points of figures and any points on the plane is an important methodological moment that teachers should prepare well. A variant can be to prepare a cardboard (or a transparent plate if it has one) where a figure and perforated points appear, inside and outside the figure. When moving the figure, the whole cardboard moves in reality, and it is not only the points on it that have their corresponding points due to the movement, but any point on the plane. This should be well emphasized.

In the previous work, the general properties of the movements must be explicit, and the following proposition must be well fixed: “A figure and its image by a movement are always equal, and if 2 figures are not equal, then none of them is the image of the other by a movement”.

The first motion that is proposed to be studied is the reflection of the plane on a straight line. The fundamental thing that the teachers must achieve when approaching this movement is that the students understand that the reflection is an example of movement in the plane and learn how the correspondence between its points is established. This should be done in close relation to the work done with axial symmetry, since reflection is nothing more than the extension of axial symmetry to any points in the plane, so that knowledge about symmetry can be used for the physical-mechanical stage of reflection.

This means that this first example of movement can be motivated by concrete situations known to the students, for example: using a mirror where an original figure appears, and its image is reflected in it: a sheet showing a landscape reflected on the waters of a river; and so on. To move on to working with models, we can use representations of symmetrical figures and take advantage of this opportunity to show that the correspondence established between symmetrical figures can be extended to all the points of the plane that contains them.

You can illustrate this correspondence with models made on the blackboard on cardboard or paper and emphasize that in this model, the cardboard (or whatever is used) represents the plane. The triangle is transformed into another triangle just like it. Each point of the original triangle has a corresponding point, and each point of the plane outside the triangle also has a corresponding point.

This correspondence is already known to schoolchildren, but limited to figures; it must now be extended to the entire plane containing the figure, and it is satisfied that:

1. Each point has exactly one image point.
2. A figure and its image are equal.

This should allow us to conclude that it is a movement of the plane that is called reflection with respect to a straight line, and with the work done with axial symmetry, the students should be able to describe how the image of each point is obtained (constructive definition), thus moving on to the stage of the constructive geometric process. It must be concluded that it is exactly the same, but in symmetry it referred only to points of the figures that were symmetrical, and here all the points of the plane are considered.

It is important in this work to emphasize that "the line of fold" in a reflection is called the axis of reflection, and that without it, it is not possible to realize it. To make them understand this necessity, you can give them one or several points on graph paper and ask them to find the image by reflection (without indicating an axis). This should provoke a contradiction because either they do not do it or each one chooses a different axis.

In the same problem, once the need for the axis has been indicated, the same situation can be posed but specifying the image of one of the points. The students must be made to understand that this information is also sufficient, since it makes it possible to determine the axis (the bisector of the segment determined by a and its image) and from it the image of the remaining points.

It is convenient to present exercises on graph paper to produce a discussion on which are the corresponding points and why, insisting on the characteristics to argue, such as the relations of equality, parallelism and perpendicularity.

It is also prudent, in order to fix the constructive definition of the reflection, to work with coordinates, which, besides contributing to fix the definition, will allow the development of skills of location in the plane by locating and identifying points by their coordinates. This has a high instructional value, inside and outside Mathematics, developing general intellectual skills such as observation.

Teachers should devise a system of exercises aimed at recognition, construction and argumentation in such a way as to help schoolchildren, while establishing the constructive definition of reflection, to continue developing their geometric thinking in exercises where they have to recognize, contribute optionally and argue on the basis of the properties studied. For this purpose, the students should actively participate in the exercise, with stages of independent work, stages of collective discussion of solutions, and, whenever necessary, the teacher's participation, clarifying and clarifying those aspects that offer doubts or have not been well concluded by the students.

In order to deal with translational motion, it is first necessary to deal with the concept of a vector. The most important thing that teachers must achieve is that their students understand the vector concept, that they can trace it, and recognize equal vectors.

The most important preconditions for the treatment of the vector concept, depending on the needs of the students, which may or may not be activated, are the position relations between straight lines and the tracing of parallel lines with ruler and bevel. For this purpose, some preliminary activities can be carried out to draw intersecting lines and parallel lines.

The vector concept has three elements that characterize it: direction, sense and length. This indicates that it is necessary to make the students have the notion of direction and sense, since they already know the length. A problem situation can be used for this purpose. If we are at the intersection of two streets and we want to find a house on one of them, what should we know?

Undoubtedly, the first answer is that we must know which of the two streets we are interested in, in other words, which of the two "directions" to choose, the direction indicated by one of the streets or the one indicated by the other street. Then the idea of "direction" can be given to us by a straight line (represented by the street), just as parallel lines define the same direction and those that intersect each other define different directions.

The second question we must ask is whether it is enough to know the street in which the house is located (the address in this case). Undoubtedly, knowing the street is not enough, since it can be on one side of the street or the other. This can be illustrated on the blackboard. Then it is also necessary to know in which "direction" it is located. Here it should be emphasized that any straight line can be traversed in two opposite directions.

These ideas of direction and sense can be reinforced with practical activities in the classroom itself: stopping two students at a time, placing them at a point in one of the aisles between the seats (the aisle will indicate the direction), and having one walk in one direction and the other in the opposite direction.

Returning to the initial problem, it is likely that the students will realize that even with the information we have, we cannot get to the house we want; it is also necessary to know how far the house is from where we are located. This can be represented on the blackboard.

In short, a direction, a sense and a length are needed. These ideas can also be illustrated in the motion of a corridor. Finally, it must be concluded that: a segment in which a direction is considered is called a vector, and therefore it has a direction, a direction and a length.

It is necessary to clearly indicate how the vectors are denoted and the importance, if they are denoted by two letters, of their order to specify the direction. You must also introduce the vector sign.

This sign should be placed on top of the 2 letters, with an arrow indicating the sense, for example:

The equality of vectors can be introduced from a worksheet, preferably on graph paper, with some vectors that are equal and others not; among those that are not equal should appear some that only have different sense. Subsequently, the observation of the students should be oriented to those that are parallel with the same direction and equal length, informing them that these vectors are equal and those that only differ in the direction are opposite vectors.

To complete the work, only the tracing of equal vectors remains. This procedure is very important for the translation movement, and to specify the steps of this process, a conversation can be promoted with the students based on a question that can be like the following one:

How would you draw a vector equal to the vector AB? Can this vector be drawn on the blackboard?

It is almost certain that you realize that you have to draw parallels and consider a length equal to the length of the vector AB. This should be used as an opportunity to make a correct trace using a ruler, a bevel, and the compass for the transport of the segment. It should be discussed how to specify the direction.

After dealing with the subject of vectors, the translation movement is dealt with. The fundamental thing that the teachers must achieve is that the students realize that the translation is an example of movement in the plane and that they learn how the correspondence between its points is established.

In order to motivate this new movement, well-known practical examples can be used, such as the movement of a vehicle on a straight street, that of a 100-meter runner moving along a straight lane, or that of a matchbox in which its interior moves, in order to open or close it, inside its outer part. This can also be illustrated by the movement of the bevel along a straight line when we want to draw parallel (or perpendicular) lines.

This movement, which is part of the mechanical physical process, can be used to introduce the term "translation" or the expression "moves" and to emphasize that the displacement, in each case, follows a given direction, a given sense and a given length.

To make it clear that translation is also a movement, a similar idea to that of reflection can be used, using a cardboard or a transparent plate (which will represent a plane). On it, any figure and perforated dots, inside and outside that figure, should appear.

It should then be noted that:

1. To each point of the plane, another point can be made to correspond.
2. The figure was transformed into another one equal to the first one.

The students are able to conclude that this is a movement and a conversation can be promoted with them so that they can suggest how, knowing the vector of the translation, the image of each point could be obtained. Another variant would be to give them the procedure. In either of the two variants, it is very important that the students realize that each point and its image determine a vector equal to that of the translation, which they can verify experimentally with their plotting instruments.

It is very appropriate to propose exercises to initiate the students in the development of skills in the recognition of corresponding points by a translation with their due documentation (that they are on parallel lines, that all the segments determined by a point and its image have the same length, and that the direction is the same).

For the construction of images, it will be pointed out to the students that the same procedure used for the construction of equal vectors is followed. The problem of the necessity of the vector that defines the translation in order to be able to perform it should be raised. In the same problem, once the need for the vector has been indicated, the same situation can be posed but specifying a point and its image. The students must be made to understand that this information is also sufficient, since it allows the vector to be determined and the image of the remaining points to be derived from it.

The exercise selected by the teachers should be directed so that the students fix the constructive definition of the translation and continue to develop their geometric thinking by solving exercises of recognition, construction and argumentation.

After the study of reflection with respect to a straight line and translation, central symmetry or reflection with respect to a point is approached as a particular case of rotation. The concept of rotation is introduced limited to the special case of central symmetry. Nevertheless, it is necessary to give some intuitive notions, within the mechanical physical process, about rotational motion in general. The fundamental thing that teachers must achieve is that schoolchildren understand that central symmetry is an example of motion in the plane, and that they learn how the correspondence between its points is established.

To motivate this new movement, examples well known to schoolchildren can be used, such as the “living uncle” and the star of the amusement park, the hands of the clock, and so on. In these examples, the term “rotation” and the points and angles of rotation should already be used. It is recommended to develop a teaching aid to illustrate these concepts; also, in a simple way, a rotation of this can be performed around a point, which remains fixed, with a certain angle.

Within this part of the mechanical physical process should be included the rotation of some object or model with an angle of 180°. For this purpose, it is very useful to use a wooden rod with a nail in its center or midpoint and a cardboard figure at one of its ends. If a line is drawn on the blackboard and the rod is placed on the line and rotated 180°, the student will observe that a rotation of 180° has been made and, in addition, how the figure image is compared to the original. This result should be used to inform the students that a rotation of 180° is called central symmetry or reflection with respect to a point, and that this is what they are going to study. In addition, the center of symmetry should be called the point of rotation, that is, the point that remains fixed.

Several examples of central symmetry should be represented with material so that the students are able to see that when this movement takes place, all the points of the plane have their corresponding points (not only those that have been highlighted) and that, in addition, the figures are transformed into others equal to them, so they are in the presence of a movement. They can also verify experimentally, using their ruler and compass, that in the central symmetry:

1. The center of symmetry is the midpoint of the segment, determined by each point and its image.
2. The segments determined by corresponding points pass through the center.

The first of these two characteristics is the basis for the constructive definition of central symmetry, so teachers can lead the process of obtaining this constructive geometric procedure, asking questions such as:

How could the image of any point P be obtained by central symmetry?

With this question, not only can we get them to give ideas of the procedure, but also to understand the need to have previously determined a center of symmetry. The students should recognize that it is necessary to know the center of symmetry to find the image of a given figure, and in turn they can be asked the following question:

How do you determine the known center of symmetry of a figure and its image?

To answer this question, there are 2 solutions that must be obtained from a dialogue with the students; they are:

1. Determine the midpoint of a segment determined by a point and its image (by tracing the perpendicular bisector).
2. Determine the intersection point of any two segments determined by corresponding points.

This last procedure is much simpler and can be used whenever at least two pairs of corresponding points are known.

The initial exercise should be aimed at getting the students to fix the properties of the movement; they should not be expected to use the constructive procedure of making the image of a figure by central symmetry (with ruler and compass), but rather, taking into account the properties of the symmetrical points and following the traces in grids, they should reproduce the image in each case.

It should also be achieved that schoolchildren, by simple inspection, recognize whether there is; they can even find the center, joining by a segment corresponding points and checking that they all pass through the same point. In all this work, it is very important that the students observe that every segment and its image by a central symmetry are parallel; this property will also be very useful to identify which movement is involved and to check the accuracy of the images they build.

After this work is done, it is convenient to exercise the construction of images by central symmetry.

It is also necessary to deal with successive movements or compositions of movements. In this respect, it is essential that the children recognize that the composition of 2 or more movements is always another movement. In particular, by means of exercises the schoolchildren should conclude that:

1. The composition of two translations is always another translation.
2. The composition of 2 reflections with respect to axes that intersect at a point is a rotation with a center at that point.
3. The composition of 2 reflections with respect to perpendicular axes is a central symmetry.
4. The composition of 2 reflections with respect to axes parallel to each other is a translation.
5. The composition of 2 central symmetries of different centers is a translation.

In summary, by means of the exercise, teachers should make the students fix the constructive definition of central symmetry and continue developing their geometric thinking by solving exercises of recognition, construction and argumentation. In particular, the students should be able to recognize, within the known plane figures, those that are symmetrical with respect to a point. In addition, it is also very important to analyze in each movement in which cases the line image of another is parallel to the original and in which cases it becomes itself.

3.6. Methodological suggestions for the treatment of the geometry of space in the second cycle of Elementary Education

Since previous grades, schoolchildren have studied the different geometric bodies. The fundamental thing is to systematize the concepts, their properties and relationships. During its introduction, the following procedure should have been taken into account:

1. Observation of geometric properties in the environment.
2. Making empirical generalizations.
3. Making theoretical generalizations.
4. Determination of other properties of the concept.
5. Search for new objects with that geometric shape in the environment.
6. Mastery and systematization of geometric skills.

It is important for teachers to have a diagnosis of schoolchildren in terms of the development of the ability to identify geometric bodies and in terms of the mastery of the essential characteristics of each one.

It is convenient to have a great variety of representatives of the bodies, differentiated by their size and color. The first thing teachers should get schoolchildren to recognize is the difference between geometric figure and geometric body:

"*All the points of a geometric figure are located in the same plane, while the same is not true for all the points of a body*".

It is convenient for schoolchildren to separate bodies into those that are limited only by flat surfaces and those that are limited by some curved surface (round bodies). In other words, the type of surface, curved or flat, is a first classification criterion.

Then we must move on to the treatment of bodies bounded only by plane figures (prisms and pyramids). In this case, schoolchildren should be guided to look for an essential difference between the representatives, and it will be possible to conclude that: some have 2 bases (prisms) and others only one (pyramid). Afterward, the concept of a prism and a pyramid is clarified, insisting on their respective characteristics. In the case of prisms, schoolchildren should remember that:

1. Prisms are geometric bodies limited by 2 parallel and equal polygons (bases) and by rectangles whose number coincides with the number of sides that the bases have.
2. There are 2 special cases: orthohedron and cube. It is important that they recognize that every cube is an orthohedron, but that there are orthohedron that are not cubes.
3. The number of edges that a prism has can be calculated by multiplying by 3 the number of sides that a base has and the number of vertices by multiplying by 2.

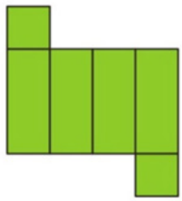
In the case of the pyramid, you must remember that they are geometric bodies limited by a polygon (base) and by triangles concurrent at a point (vertex) whose number coincides with the number of sides that the base has. The number of edges of a pyramid is obtained by multiplying by 2 the number of sides of the base and the number of vertices by adding 1 to the number of vertices of the base. You may ask: What is the base of a prism that has a total of 6 vertices? How many edges does it have? Successfully answering this question requires extensive work on the concrete material.

When dealing with round bodies, it should be noted that some are bounded by both plane figures and curves (cylinders and cones) and others only by a curve (sphere). When working with the cylinder, it is important to emphasize that it is limited by two flat surfaces that are circles as bases, and the rest is a curved surface.

After cone-shaped objects of the medium are presented, it is considered important that in models the students can emphasize its difference with the cylinder: it has only one flat surface that is also a circle as a base. In the treatment of the sphere, it should be emphasized that, unlike prisms and pyramids, it has no bases or faces and is completely formed by curved surfaces.

Another very important aspect to work on is the development of the bodies. Exercises should be presented so that the students can identify the development of a given body.

Example:



Determine to which geometric body the following development corresponds.

It is also productive for them to learn how to construct the development of bodies and assemble them, to identify faces from different views in both simple and compound bodies.

3.6.1. The treatment of the volume of the orthohedron

Special attention deserves the treatment of the volume of the orthohedron. The fundamental thing in this sense is that the schoolchildren have a mental representation of what is the volume of an orthohedron and can transfer it to the volume of the remaining bodies. The most important preconditions are the length of a segment and the area of the rectangle, since the volume is worked out similarly. It is recalled that the area of a rectangle was made by counting a unit square of 1 cm^2.

It is of vital importance that schoolchildren master that volume is the place occupied by a body in three-dimensional space, since all objects have volume. Schoolchildren should reach these conclusions after analyzing environmental situations proposed by teachers.

To determine the volume of the orthohedron, it is recommended that the teachers present a unit cube with a side of 1 cm and several orthohedra (previously analyzed that the unit cube fits an exact number of times in the orthohedra). We suggest that they be orthohedra with the following dimensions:

Orthohedron 1: 2. 3. 4

Orthohedron 2: 1. 6. 4

Steps to follow:

1. Creation of a problem situation:

You know how to calculate the perimeter of a polygon and the area of a polygon, especially that of a rectangle. How can you measure the “extent of the orthohedron” or “what fits inside an orthohedron”? At this point, you can take the opportunity to inform them that the above is called the volume of the orthohedron.

2. The unit cube is shown:

Teachers show the students a cube (one can be given per team) and ask them to measure its sides; they will check that it measures 1 cm on a side. They are informed that this unit cube represents a cubic centimeter, and they write 1 cm^3.

3. The unit cube is compared with several orthohedra:

The teachers inform that now it is a question of finding out the volume of the orthohedra presented initially. Through a conversation, the students will be able to answer by analogy that we have to see how many times the unit cube fits in each orthohedron. By comparing, it will be possible to conclude that it fits 24 times in each one.

4. Final details:

Teachers report that the volume of each orthohedron is 24 cm^3, and write V = 24 cm^3. Then the students are asked: How can the volume of the orthohedra be calculated without comparing with the unit cube of 1 cm^3, knowing only the dimensions of length, width and height? We can conclude that by multiplying the 3 dimensions: a. b. c. For orthohedron 1 we have V = 2 cm. 3cm .4cm = 24 cm^3 and for the second V = 1 cm. 6 cm. 4 cm = 24 cm^3. It is also worth noting that the orthohedra do not have the same dimensions, and yet they have the same volume.

After elaborating the volume of the orthohedron, we must continue with the treatment of the remaining units of volume. Subsequently, one should analyze the remaining multiples and submultiples of the cubic meter and conclude that:

"Each unit is 1 000 times larger than the one immediately below it and 1 000 smaller than the one immediately above it".

From the above, it can be inferred that when you want to go from a larger unit to the next lower one, you have to multiply, in this case, by 1,000 and from a smaller one to the next higher one, you have to divide, in this case, by 1,000. This is very important for conversions. Here they can also make use of a means in the form of a ladder, which, as with the units of length and area, will facilitate the conversions, so it is convenient for the students to memorize it.

In order to memorize the multiples and submultiples of the cubic meter (m3), oral exercises can be done to convert from a larger unit to a smaller one and vice versa. Then you can give any unit and ask for the one that follows it, or the one before it, or all the ones that follow it, or that are before it.

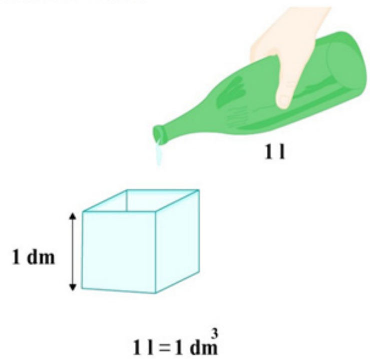
A concept that is closely related to that of volume is that of capacity. Capacity is the quantity that fits in an object or container. It is considered necessary for schoolchildren, starting from concrete objects, to be able to know that all bodies have volume but not all have capacity. The fundamental unit of capacity is the liter (l). In liters, we can express the amount of water that a swimming pool can contain, the amount of gasoline that fits in the tank of a gas station, the amount of oil that fits in a knob, and so on.

To work with the multiples and submultiples of the liter, proceed in the same way as with the multiples and submultiples of the cubic meter. It is considered necessary to present the following problem situation:

How many liters of water fit into a cubic-shaped container that is 1 dm on a side?

It is in this analysis where it will be possible for the students to understand the equivalence relationship between the units of volume and the units of capacity by arriving at the conclusion that

That a liter (l) is the capacity of a cubic container of 1 dm of side: 1 l = 1 dm^3.



3.7. The treatment of graphs

The treatment of graphs plays a very important role, which is also addressed in other branches of mathematics. The important thing to achieve is that schoolchildren learn to interpret graphs and recognize their usefulness in representing and understanding situations in different spheres of life. The graph is the geometric representation of numerical data by means of lines, points, rectangles, and etcetera.

However, it is of interest that schoolchildren learn how to construct graphs. The following are some requirements that should be addressed by teachers so that schoolchildren acquire certain skills in graph making.

To motivate them, they can be shown pictures in which practical situations are represented by bar graphs, line graphs or pie charts.

In bar charts:

1. The bars are at the same distance from each other on the x-axis, and may also be next to each other.
2. The height of each bar depends on the data to which it corresponds and is sought on the y-axis.
3. On the y-axis, the divisions are made to scale.

In pie charts:

1. Pie charts are used to represent situations referring to the same quantity.
2. The circle is used for this representation. The center of the circle is the vertex of an angle of 360^0.
3. To construct a pie chart, it is very important to apply the knowledge about the relationship between the parts and the whole, when representing fractions in a circle. That is, divide the circle into equal parts and take some of those parts. The sum of all the fractions represented in the graph is equal to 1. If the data are expressed in percent, the sum must be equal to 100 %.

In line graphs:

1. The way in which each point is determined is the same as in the bar graph, but they are joined by segments.

As a practical activity, the students will be able to make graphs related to the economic and social development of the context in which they live.

Supplementary materials

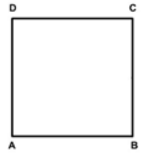
1. Procedure for the construction of a tangram

The tangram or game of the seven elements is a puzzle, which has its origins in ancient Chinese culture; it consists of 7 pieces (5 triangles, 1 square and 1 parallelogram), obtained from a square. It can be used to represent objects, moving silhouettes and geometric figures.

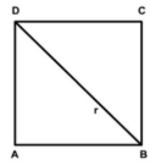
The following is a procedure that you can follow to make a tangram:

Materials needed: cardboard, ruler, bevel, colors or tempera and scissors.

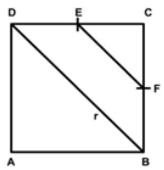
1. Draw on the cardboard a square of 20 cm on each side and denote it with the letters A, B, C and D.



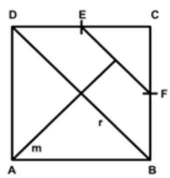
2. Then draw a straight line r, with the ruler, in such a way that it divides that square into two equal triangles.



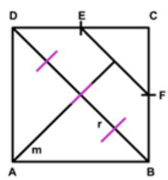
3. The points E and F, respectively, are located in the middle of the segments DC and BC, and a segment is drawn joining these points and parallel to the line r.



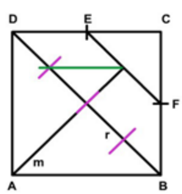
4. Draw a line m, perpendicular to the segment EF, in such a way that it cuts the line r and passes through the point A.



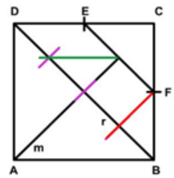
5. Divide the line r, with the ruler, in four equal parts.



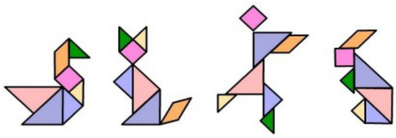
6. Draw the green line shown below, parallel to the segment DE.



7. Finally, draw this red line perpendicular to line r and cut out each of the pieces that make it up.



Some figures that can be assembled with the tangram:



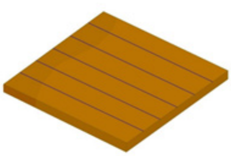
2. Procedure for the construction of a geoplane

Geometric figures and movements can be represented with rubber bands on the geoplane.

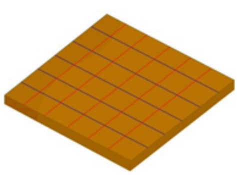
The following is a procedure for its construction:

Materials needed: Piece of wood, a pencil, a ruler, a ruler, a bevel, twenty-five nails and a hammer.

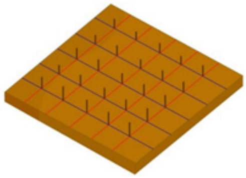
1. On a board where one of its sides has 30 cm sides, divide its left side into 6 equal parts and draw five parallel straight lines.



2. Then divide the upper part of the board into six equal parts and draw five parallel lines that cut the previous lines perpendicularly and that form squares.



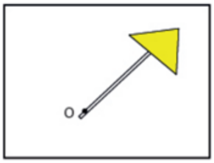
3. Finally, place each of the nails in the place where the lines are cut.



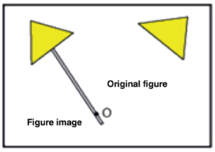
3. Procedure for the construction of a teaching aid for rotational motion

Materials needed: Cardboard, a pencil, ruler or bevel, nail, colors or tempera.

1. A triangle is cut out of cardboard equal to the trace and glued to a rectangle that serves as a frame, to which a lace can be attached, so that the triangle can be rotated around it.



2. Mark the initial position (original figure) and the mobile triangle in its final position (figure image).

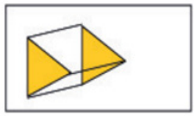


4. Procedure for the construction of a teaching aid for translation

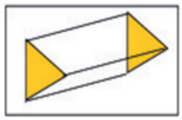
Materials needed: Cardboard, a pencil, ruler or bevel, nail, colors or tempera.

1. Cardboard with a fixed triangle and a movable triangle with pieces of yarn of equal length, one at each vertex, which are located behind it so that they are not visible.

2. The movable triangle is moved as shown on the sheet.



3. Move the movable triangle according to the length of the threads and fix the triangle in its final position (picture).



REFERENCES

Alsina Catalá, C., Burgués Flamarich, C., & Fortuny Aymemí, J. M. (1989). Invitación a la Didáctica de la Geometría. Síntesis, S.A.

Alsina Catalá, C., Burgués Flamarich, C., & Fortuny Aymemí, J. M. (1991). Materiales para construir la Geometría. Síntesis, S.A.

Álvarez de Zayas, C. M. (1989). Fundamentos teóricos de la dirección del proceso docente educativo en la Educación Superior cubana. MES.

Álvarez de Zayas, C. M. (1999). La escuela en la vida. Pueblo y Educación.

Ballester Pedroso, S., et al. (2001). Metodología de la enseñanza de la Matemática I. Pueblo y Educación.

Barcia Martínez, R. (2000). La preparación geométrica de los Licenciados en Educación Primaria. (Doctoral Thesis). Universidad "Carlos Rafael Rodríguez".

Brihuega Nieto, J. (2006). Espacio y forma. Materiales para el aula. http://www.galega.org/emdg/web/Espacio%20_y%20forma.doc

Bronzina, L., Chemello, G., & Agrasar, M. (2009). Segundo Estudio Regional Comparativo y Explicativo. Aportes para la enseñanza de la Matemática. Salesianos Impresores S.A.

Cajaraville Pegito, J. A. (1989). Ordenador y Educación Matemática. Algunas modalidades de su uso. Síntesis.

Calvo Penadés, X., Carbó Colomer, M. C., Farell Pastor, M., & Fortuny Aymemi, J. M. (2002). La geometría: de las ideas del espacio al espacio de las ideas en el aula. Grao.

Campistrous, L., & Rizo, C. (1996). Aprende a resolver problemas aritméticos. Pueblo y Educación.

Canals Tolosa, M. A. (1997). La Geometría en las primeras edades escolares. Revista Suma, 25, 1-34.

Castellanos Simons, D., Castellanos Simons, B., Llivina Lavigne, M. J., Silverio Gómez, M., Reinoso Cápiro, C., & García Sánchez, C. (2002). Aprender y enseñar en la escuela. Una concepción desarrolladora. Pueblo y Educación.

Castillo, S., Arrieta, L., & Rodríguez, M. E. (2006). Epistemología y método en Educación Matemática. Copérnico: Revista Arbitrada Interdisciplinaria, 4, 51-58.

Chevallard, Y. (2005). La transposición didáctica. Del saber sabio al saber enseñado. Aique Grupo Editor, SA.

Cruz Ramírez, M. (2006). La enseñanza de la Matemática a través de la resolución de problemas. Educación cubana.

D'Ambrosio, U. (2005). La integración de la matemática con las ciencias. Matematicalia: Revista digital de divulgación matemática, 1(1).

De Guzmán, M. (1993). Tendencias innovadoras en Educación Matemática. Organización de Estados para la Educación y la Cultura. Popular.

Díaz Godino, J. (2010). Perspectiva de la didáctica de las matemáticas como disciplina tecnocientífica. Universidad de Granada. http://www.ugr.es/local/jgodino

Díaz Godino, J., & Batanero, C. (2009). Formación de profesores de matemáticas basada en la reflexión guiada sobre la práctica. Universidad de Granada. http://www.ugr.es/local/jgodino

Díaz Godino, J., & Ruíz, F. (2002). Geometría y su didáctica para maestros. ReproDigital.

Díaz Godino, J., Batanero, C., & Font, V. (2004). Didáctica de las matemáticas para maestros. GAMI, S. L. Fotocopias.

Fridman, L.M. (1977). Análisis lógico-psicológicos de los problemas docentes. Editorial Pedagógica.

Galindo, C. (1996). Desarrollo de habilidades básicas para la comprensión de la geometría. Revista EMA, 1, 49-58.

Galperin, P. Y. (1982). Introducción a la Psicología. Pueblo y Educación.

Geissler, OStR E, et al. (1977). Metodología de la enseñanza de la Matemática 1 a 4. Grado. First part. Pueblo y Educación.

Geissler, OStR E, et al. (1978). Metodología de la enseñanza de la Matemática 1 a 4. Grado. Third part. Pueblo y Educación.

Giorgion, R. (2010). Habilidades matemáticas presentes em alunos do ensino medio participantes em feiras de ciências. (Mestrado em Educação Matemática). Pontífícia Universidade Católica.

Jaime Pastor, A., & Gutiérrez Rodríguez, Á. (1996). El grupo de las isometrías del plano. Síntesis.

Junquera Muné, J. (1961). Didáctica del cálculo. Labor.

Konstantinov, F., Malinin, V., Sorokoúmskaya, N., Ermolaéva, V., & Latínskaya, T. (1980). Fundamentos de filosofía marxista - leninista. First part. Ciencias Sociales.

Labarrere Reyes, G., & Valdivia Pairol, G. (1988). Pedagogía. Pueblo y Educación.

Labarrere, A. (1987). Bases psicopedagógicas de la enseñanza de la solución de problemas matemáticos en la escuela primaria. Editorial Pueblo y Educación.

Lenin, V. I. (1964). Obras Completas. Volume 29. Editora Política.

León González, J. L. (2011). Estrategia didáctica para el desarrollo de habilidades geométricas en el primer ciclo de la Educación Primaria. (Doctoral Thesis). Universidad de Ciencias Pedagógicas: "Conrado Benítez García".

León González, J. L., & Barcia Martínez, R. (2006). La habilidad de reconocimiento geométrico de figuras compuestas en los escolares primarios. Conrado, 2(6).

León González, J. L., & Barcia Martínez, R. (2010). Propuesta para la elaboración y utilización del tangram y el geoplano en el proceso de enseñanza-aprendizaje de la geometría de la Educación Primaria. Conrado, 6 (25).

León Roldán, T. (2007). Concepción didáctica para la enseñanza y el aprendizaje de la geometría con un enfoque dinámico en la Educación Primaria. (Doctoral Thesis). Instituto Central de Ciencias Pedagógicas.

Majmutov, M. I. (1983). La enseñanza problémica. Pueblo y Educación.

National Council of Teachers of Matemathics. (2000). Principios para matemáticas escolares. NCTM. http://www.eduteka. org/PrincipiosMath.php

Ortiz Ocaña, A. L. (2006). Diccionario pedagógico, didáctico y metodológico. Hacia una pedagogía integradora y científica. CEPEDID.

Panizza, M. (2003). Conceptos básicos de la teoría de las situaciones didácticas. http://www.librospdf.net/panizzamabell-/1/

Petrovski, A. V. (1986). Psicología General. Libros para la Educación.

Polya, G. (1976). ¿Cómo plantear o resolver problemas? Trillas.

Ponce Solozábal, J. R. (1988). El sistema psíquico del hombre. Editorial Científico-Técnica.

Proenza Garrido, Y. (2002). Modelo didáctico para el aprendizaje de los conceptos y procedimientos geométricos en la escuela primaria. (Doctoral Thesis). ISP "José de la Luz y Caballero.

Puig Unzueta, S. (2003). Una aproximación a los niveles de desempeño cognitivo en los alumnos. Instituto Central de Ciencias Pedagógicas.

Rico Montero, P., Santos Palma, E. M., & Martín-Viaña Cuervo, V. (2008). Exigencias del modelo de escuela primaria para la dirección por el maestro de los procesos de educación, enseñanza y aprendizaje. Pueblo y Educación.

Rizo Cabrera, C. (1987). Estructuración del curso de geometría de cuarto a sexto grados basados en las transformaciones y la congruencia. (Doctoral Thesis). Instituto Central de Ciencias Pedagógicas.

Rosental, M., & Ludin, M. (1981). Diccionario Filosófico. Instituto Cubano del Libro.

Rubinstein, S. L. (1966). El proceso del pensamiento. Editorial Universitaria.

Ruiz de Ugarrio, G. (1965). Cómo enseñar Aritmética en la escuela primaria. Pedagógica.

Ruiz Espín, L., et al. (2000). Metodología de la Educación Plástica en la Edad Infantil. Pueblo y Educación.

Savin, N. V. (1972). Pedagogía. Pueblo y Educación.

Silvestre Oramas, M., & Zilberstein Toruncha, J. (2002). Hacia una didáctica desarrolladora. Pueblo y Educación.

Talízina, N. (1988). Psicología de la enseñanza. Progreso.

Van Hiele, P. M. (1957). El problema de la comprensión: En conexión con la comprensión de los escolares en el aprendizaje de la geometría. (Doctoral Thesis). Universidad Real de Utrecht.

Vigotsky, L.S. (1979). El desarrollo de los procesos psicológicos superiores. Crítica.

Wielewski, G. D. (2005). Aspectos do pensamento matemático na resolução de problemas: uma apresentação contextualizada da obra de Krutetskii. (Doctoral Thesis). Pontíf́ícia Universidade Católica.

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In the teaching-learning process of Mathematics in Elementary Education, one of the main difficulties is related to the development of geometric skills in schoolchildren. For this reason, the book focuses the study on geometric skills: recognizing geometric objects; tracing and/or constructing; arguing geometric propositions; and solving geometric calculation problems. The book provides the theoretical foundations for the development of geometric skills in elementary education: principles, actions, operations, levels and indicators. It also includes methodological recommendations for the treatment of the fundamental geometric objects that are addressed at this educational level, separated by cycle, due to the way in which the contents are treated in each of the grades. Teachers are also provided with the procedure to carry out some teaching aids for the treatment of geometric contents and the development of skills.

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